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A novel time continuous *p*-Galerkin (TCG) scheme with 2p-order accuracy for seismic dynamic analysis

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Abstract

A novel time continuous *p*-Galerkin (TCG) scheme for seismic dynamic analysis is proposed. Based on the new variational expression transformed from integration of the standard weak form of weighted-residual-method by parts, a new recurrence relation is derived. The convergence order of the of the displacement and velocity calculated by the derived recurrence relation is up to 2*p*, which is twice that of the standard TCG method and is one order higher than that of the standard time-discontinuous Galerkin (TDG) method at the same number of degrees of freedom. Examples of a SDOF and a Multi-DOF system are given to verify the accuracy of new TCG scheme. Simple numerical tests show a significant reduction in the computation time for the new TCG scheme in comparison to that for the Newmark method.

Keywords: time continuous p-Galerkin, linear elasto-dynamics, 2p-order accuracy, weighted-residual-method

1. Introduction

Numerical simulation of structural dynamic response is frequently used in structural seismic analysis and engineering seismic research. The core problem of numerical simulation is to solve a system of Ordinary Differential Equations (ODE) in time which are derived from the application of finite elements in space. The research of the methods for solving the ODE is also a hotspot [1]-[4] . Most of these methods are second-order accurate methods, such as the Newmark method, the Wilson method, and the HHT method. For the second-order algorithm, accuracy and efficiency is difficult to further improve. Therefore, higher order algorithms are proposed successively.

Zienkiewicz [5] have developed a set of algorithms, which is the unified set of a single step method, based on the application of the weighted residual method to the derived ODE. Many method for solving the derived ODE mentioned above are particular cases of the unified set. High accuracy can be obtained by using higher-order interpolation polynomials. Recently, new high-order accurate methods with a step-by-step time integration scheme have been developed for dynamics analysis [6]-[12], among which, the Time Continuous p-Galerkin (TCG) methods and Time Discontinuous p-Galerkin (TDG) methods are most representative. The TCG and TDG methods improve the accuracy of the solution, but also resulted in a lower solution efficiency. This paper presents a new TCG method with higher accuracy and efficiency.

The focus of this paper is the development of a novel TCG scheme for seismic dynamics analysis. Based on the new variational expression transformed from integration of the standard weak form of weighted-residual-method by parts, a new recurrence relation is derived. The convergence order of the of the displacement and velocity calculated by the derived recurrence relation is up to 2p, which is twice that of the standard TCG method [20] and is one order higher than that of the standard TDG method at the same number of degrees of freedom. Examples of a SDOF and a Multi-DOF system are given to verify the accuracy of new TCG scheme. Simple numerical tests show a significant reduction in the computation time for the new TCG scheme in comparison to that for the Newmark method.

2. Model problem and the new TCG scheme

In structural dynamic analysis, one always need to solve the system of second-order ordinary differential equations of the form

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{F}(t) \qquad t\in I=(0,T) \tag{1.}$$

with initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0,\ \dot{\mathbf{u}}(0)=\mathbf{v}(0)=\mathbf{v}_0 \tag{2.}$$

Where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{F}$ is the vector of applied forces, $\mathbf{u}$, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ are the displacement, velocity and acceleration vectors, respectively, I is the time domain, and $\mathbf{u}_0$ and $\mathbf{v}_0$ are the given initial values. The dimension of the system is denoted by n_{eq}.

To solve the model problem, the Galerkin method is considered. Let $0=t_0<t_1<\cdots<t_N=T$ be a partition of the time domain $I=(0,T)$ with corresponding time steps $\Delta t_n=t_n-t_{n-1}$ and $I_n=(t_{n-1},t_n)$.

Once the initial displacement $\mathbf{u}_{n-1}$ and velocity $\mathbf{v}_{n-1}$ of the domain $I_n=(t_{n-1},t_n)$ is got, the displacement $\mathbf{u}_n$ and velocity $\mathbf{v}_n$ can be solved by the new TCG scheme which are the initial values of the next time domain $I_n=(t_n,t_{n+1})$. The TCG procedures are as follows.

To solve the Eq. (1), the weighted residual method is used as Eq. (3)

$$\int_{t_{n-1}}^{t_n}\mathbf{w}^T(\mathbf{M}\ddot{\mathbf{u}}+\mathbf{C}\dot{\mathbf{u}}+\mathbf{K}\mathbf{u})\mathrm{d}t=\int_{t_{n-1}}^{t_n}\mathbf{w}^T\mathbf{F}\mathrm{d}t \tag{3.}$$

the Eq. (3) can be transform to Eq. (4) by partial integration.

$$\mathbf{w}^T\mathbf{M}\dot{\mathbf{u}}\Big|_{t=t_n} - \mathbf{w}^T\mathbf{M}\dot{\mathbf{u}}\Big\|_{t=t_{n-1}} + \int_{t_{n-1}}^{t_n}(-\dot{\mathbf{w}}^T\mathbf{M}\dot{\mathbf{u}} + \mathbf{w}^T\mathbf{C}\dot{\mathbf{u}} + \mathbf{w}^T\mathbf{K}\mathbf{u})\mathrm{d}t = \int_{t_{n-1}}^{t_n}\mathbf{w}^T\mathbf{F}\mathrm{d}t \tag{4.}$$

where

$$\mathbf{w} = \mathbf{N}\overline{\mathbf{w}}, \quad \mathbf{u} = \mathbf{N}\overline{\mathbf{u}} \tag{5.}$$

Where $\mathbf{N}$ are the finite element basis functions denoted by

$$\mathbf{N} = \left[\mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{N}_{p+1}\right], \quad \overline{\mathbf{w}} = \left[\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2, \cdots, \overline{\mathbf{w}}_{p+1}\right]^T, \quad \overline{\mathbf{u}} = \left[\overline{\mathbf{u}}_1, \overline{\mathbf{u}}_2, \cdots, \overline{\mathbf{u}}_{p+1}\right]^T \tag{6.}$$

where

$$\mathbf{N}_j = N_j\mathbf{I}, \quad \overline{\mathbf{w}}_j = (\overline{w}_j^1, \overline{w}_j^2, \cdots)^T, \quad \overline{\mathbf{u}}_j = (\overline{u}_j^1, \overline{u}_j^2, \cdots)^T, \quad (j = 1, 2, \cdots, p+1) \tag{7.}$$

Where N_j are the j-th Lagrange shape function.

$$\overline{\mathbf{w}}^T\mathbf{M}\dot{\mathbf{u}}\Big|_{t=t_n} - \overline{\mathbf{w}}^T\mathbf{M}\dot{\mathbf{u}}\Big|_{t=t_{n-1}} + \overline{\mathbf{w}}\int_{t_{n-1}}^{t_n}(-\dot{\mathbf{N}}^T\mathbf{M}\dot{\mathbf{N}} + \mathbf{N}^T\mathbf{C}\dot{\mathbf{N}} + \mathbf{N}^T\mathbf{K}\mathbf{N})\overline{\mathbf{u}}\mathrm{d}t = \overline{\mathbf{w}}\int_{t_{n-1}}^{t_n}\mathbf{N}^T\mathbf{F}\mathrm{d}t \tag{8.}$$

And since $\overline{\mathbf{w}}$ is arbitrary we have a set of equations which is sufficient to solve the parameter $\overline{\mathbf{u}}$ as

$$\mathbf{G}\overline{\mathbf{u}} = \overline{\mathbf{F}} + \begin{bmatrix} \mathbf{M}\dot{\mathbf{u}}_{n-1} \\ \vdots \\ -\mathbf{M}\dot{\mathbf{u}}_n \end{bmatrix} \tag{9.}$$

where the matrix $\mathbf{G}$ is denoted as

$$\mathbf{G} = -\overline{\mathbf{M}} + \overline{\mathbf{C}} + \overline{\mathbf{K}} \tag{10.}$$

and where

$$\begin{aligned} \overline{\mathbf{M}} &= \int_{t_{n-1}}^{t_n}\dot{\mathbf{N}}^T\mathbf{M}\dot{\mathbf{N}}\mathrm{d}t \qquad & \overline{\mathbf{C}} &= \int_{t_{n-1}}^{t_n}\mathbf{N}^T\mathbf{M}\dot{\mathbf{N}}\mathrm{d}t \\ \overline{\mathbf{K}} &= \int_{t_{n-1}}^{t_n}\mathbf{N}^T\mathbf{M}\mathbf{N}\mathrm{d}t \qquad & \overline{\mathbf{F}} &= \int_{t_{n-1}}^{t_n}\mathbf{N}^T\mathbf{F}\mathrm{d}t \end{aligned} \tag{11.}$$

when $\overline{\mathbf{u}}$ and $\overline{\mathbf{u}}_{p+1}$ are

$$\begin{bmatrix} \mathbf{G}_{1,1} & \mathbf{G}_{1,2} & \cdots & \mathbf{G}_{1,p+1} \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} & \cdots & \mathbf{G}_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{p+1,1} & \mathbf{G}_{p+1,2} & \cdots & \mathbf{G}_{p+1,p+1} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{u}}_1 \\ \overline{\mathbf{u}}_2 \\ \vdots \\ \overline{\mathbf{u}}_{p+1} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{F}}_1 \\ \overline{\mathbf{F}}_2 \\ \vdots \\ \overline{\mathbf{F}}_{p+1} \end{bmatrix} + \begin{bmatrix} \mathbf{M}_{1,1}\dot{\overline{\mathbf{u}}}_1 \\ 0 \\ \vdots \\ -\mathbf{M}_{p+1,p+1}\dot{\overline{\mathbf{u}}}_{p+1} \end{bmatrix} \tag{12.}$$

which is equivalent to

$$\begin{bmatrix} \mathbf{G}_{1,2} & \mathbf{G}_{1,3} & \cdots & \mathbf{G}_{1,p+1} \\ \mathbf{G}_{2,2} & \mathbf{G}_{2,3} & \cdots & \mathbf{G}_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{p,2} & \mathbf{G}_{p,3} & \cdots & \mathbf{G}_{p,p+1} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{u}}_2 \\ \overline{\mathbf{u}}_3 \\ \vdots \\ \overline{\mathbf{u}}_{p+1} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{F}}_1 - \mathbf{G}_{1,1}\overline{\mathbf{u}}_1 \\ \overline{\mathbf{F}}_2 - \mathbf{G}_{2,1}\overline{\mathbf{u}}_1 \\ \vdots \\ \overline{\mathbf{F}}_p - \mathbf{G}_{p.1}\overline{\mathbf{u}}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{M}_{1,1}\dot{\overline{\mathbf{u}}}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{13.}$$

$$\mathbf{M}_{p+1,p+1}\dot{\overline{\mathbf{u}}}_{p+1} = \overline{\mathbf{F}}_{p+1} - \mathbf{G}_{p+1,j}\overline{\mathbf{u}}_j \tag{14.}$$

It's obvious that the implementation of the method is indeed not so complex, though the computation effort needed within each time step is still high than the commonly used one-step method Newmark scheme. Let us estimate the computational cost in a rough way.

3. NUMERICAL EXAMPLES

Example 1: SDOF model

Consider the following simple problem.

$$\begin{cases} \ddot{u} + 0.1\dot{u} + u = \sin(0.5t) \\ u(0) = 0, \quad v(0) = 0 \end{cases} \tag{15.}$$

Where u, $\dot{u}$ and $\ddot{u}$ represent the displacement, velocity and acceleration respectively. This is the example being used to illustrate the DG method by Li et al [23].

The problem is solved by the Newmark method and the TCG method and their errors are compared in Fig.1. u^N denotes the Newmark solutions and u^h denotes the solutions of TCG method. It is shown that for element of degree *p* with sufficient smooth solutions, the displacement, velocity and acceleration of the scheme converge at the order of 2p while the Newmark method converge at the order of 2.

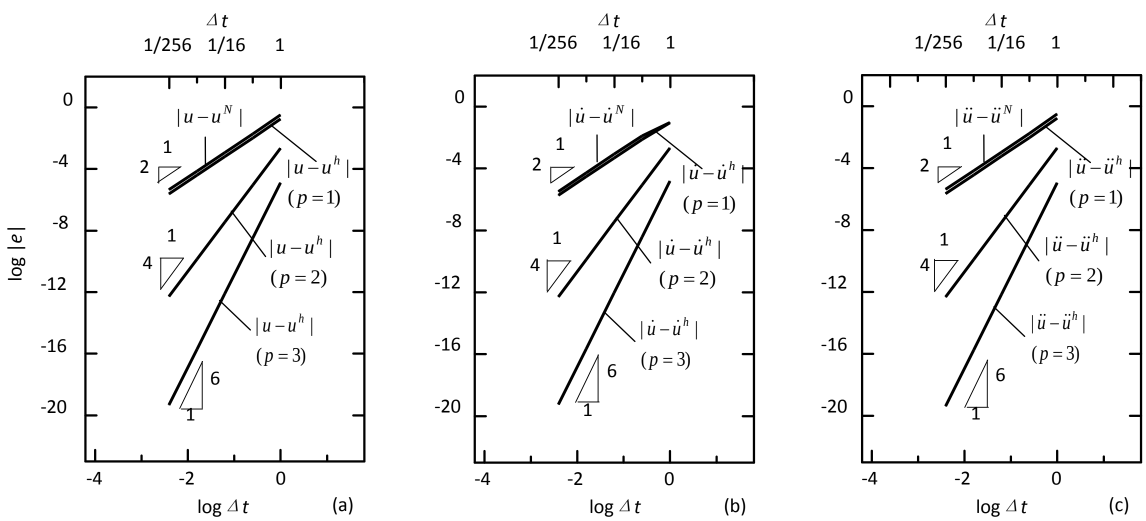

Fig.1 Convergent rate of the Newmark method and the new TCG method of example 1:

(a) Displacement, (b) Velocity, (c) Acceleration.

Example 2: Multi-DOF model

Consider the following two-DOF system..

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{16.}$$

Assume $m_1 = 2$, $m_2 = 1$, $k_{11} = 6$, $k_{12} = k_{21} = -2$, $k_{22} = 4$, $f_1 = 0$, $f_2 = 10$ and zero initial conditions. This is the example being used to illustrate various direct integration methods by bathe [23].

The problem is solved by the Newmark method and the TCG method and their errors are compared in Fig.2. u^N denotes the Newmark solutions and u^h denotes the solutions of TCG method. It is shown that for element of degree *p* with sufficient smooth solutions, the displacement, velocity and acceleration of the scheme converge at the order of 2*p* while the newmark method converge at the order of 2.

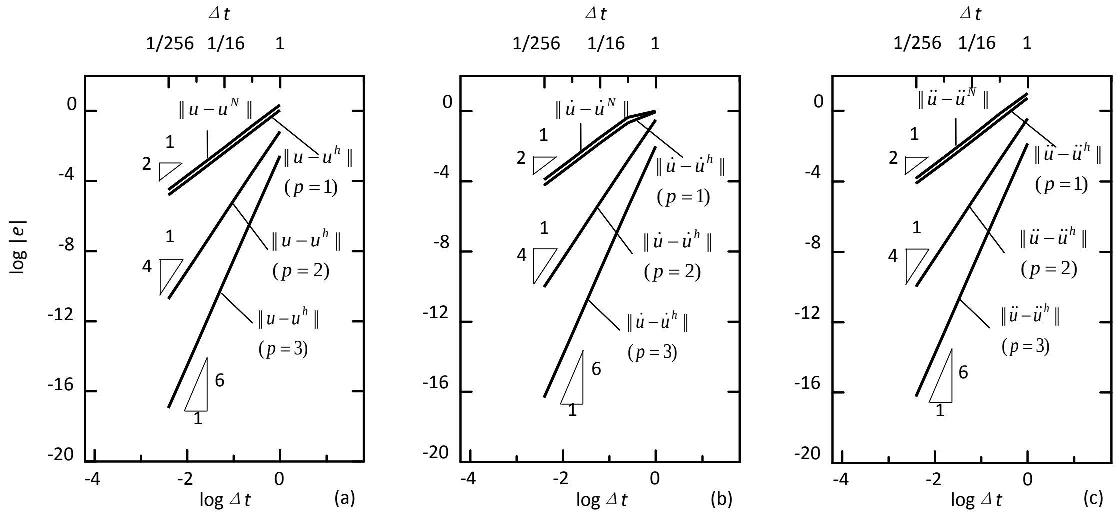

Fig.2 Convergent rate of the Newmark method and the new TCG method of example 2:

(a) Displacement, (b) Velocity, (c) Acceleration.

4. Concluding remarks

A new high-order accurate TCG method for seismic dynamic analysis is suggested. The comparison of the New TCG method with standard TCG and standard TDG is shown in Table 1. The main advantages of the new method are the higher order of accuracy in comparison to that of known high-order accurate methods (if the same number of degrees of freedom is used) and controllable numerical dissipation.

Table 1 Accuracy of TCG and TDG methods for elastodynamics

Number of degrees of freedom	New TCG		Standard TCG		Standard TDG	
	Order of accuracy in time	Time approximations	Order of accuracy in time	Time approximations	Order of accuracy in time	Time approximations
n_{eq}	2	$p=1$	1	$p=1$		
$2n_{eq}$	4	$p=2$	2	$p=2$	3	$p=1$
$3n_{eq}$	6	$p=3$	3	$p=3$	5	$p=2$

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References

[1] Hilber, H M, & Hughes, T J (1978). Collocation, dissipation and [overshoot] for time integration schemes in structural dynamics. Earthquake Engineering & Structural Dynamics, 6(1), 99-117.

[2] Hilber, H M, Hughes, T J, & Taylor, R L (1977). Improved numerical dissipation for time integration algorithms in structural dynamics. Earthquake Engineering & Structural Dynamics, 5(3), 283-292.

[3] Newmark, N M (1959): A method of computation for structural dynamics. *Journal of the Engineering Mechanics Division*, 85(3), 67-94.

[4] Wilson, E L, Farhoomand, I, & Bathe, K J (1972): Nonlinear dynamic analysis of complex structures. *Earthquake Engineering & Structural Dynamics*, 1(3), 241-252.

[5] Zienkiewicz, O C, Wood, W L, Hine, N W, & Taylor, R L (1984): A unified set of single step algorithms. Part 1: General formulation and applications. *International Journal for Numerical Methods in Engineering*, 20(8), 1529-1552.

[6] Bonelli, A, Bursi, O S, & Mancuso, M (2001): Explicit predictor–multicorrector time discontinuous Galerkin methods for linear dynamics. *Journal of Sound and Vibration*, 246(4), 625-652.

[7] Bonelli, A, & Bursi, O S (2003): Iterative solutions for implicit time discontinuous galerkin methods applied to non-linear elastodynamics. *Computational mechanics*, 30(5-6), 487-498.

[8] Fung, T C (2005): Construction of higher-order accurate time-step integration algorithms by equal-order polynomial projection. *Journal of Vibration and Control*, 11(1), 19-49.

[9] Kunthong, P, & Thompson, L L (2005): An efficient solver for the high-order accurate time-discontinuous Galerkin (TDG) method for second-order hyperbolic systems. *Finite Elements in Analysis and Design*, 41(7), 729-762.

[10] Mancuso, M, & Ubertini, F (2003): A methodology for the generation of low ‐ cost higher ‐ order methods for linear dynamics. *International journal for numerical methods in engineering*, 56(13), 1883-1912.

[11] Thompson, L L, & He, D (2005): Adaptive space–time finite element methods for the wave equation on unbounded domains. *Computer methods in applied mechanics and engineering*, 194(18), 1947-2000.

[12] Wiberg, N E, & Li, X (1999): Adaptive finite element procedures for linear and non ‐ linear dynamics. *International journal for numerical methods in engineering*, 46(10), 1781-1802.

[13] Li, X D, & Wiberg, N E (1996): Structural Dynamic Analysis By A Time-Discontinuous Galerkin Finite Element Method. *International Journal for Numerical Methods in Engineering*, 39(12), 2131-2152.

[14] Bath, K J (1982): Finite element procedures in engineering analysis. Pentice-Hall, Englewood Cliffs, NJ.