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Simplified Numerical Model for Seismic Collapse of RC Frame Structures

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Abstract

Collapse of buildings due to an earthquake is completely unacceptable. The combination of wide usage of Reinforced Concrete (RC) framed structures for residential buildings in earthquake prone areas and shoddy design and construction practices exposes the high vulnerability of this type of buildings to a seismic hazard. For practical engineering purposes, the following characteristics are desirable in a collapse simulation, (i) capturing the complete behavior of a structure i.e. from its normal state to a complete collapsed state during an earthquake, (ii) should be accurate and reliable, (iii) computationally efficient, (iv) ease of modelling, comprehending and workability, and, (v) good visualization of the analysis results.

Considering the above requirements, an effective two-phase numerical collapse simulation model of structures is proposed. The first phase consists of a Finite Element Mapped Spring Network which simulates the small deformation nonlinear behavior of the RC frames. In the second phase, the aforementioned spring network is adopted into the Extended Distinct Element Method (EDEM) to predict the large deformation behavior of structures till collapse.

To model concrete non-linearity, spatially averaged material models for concrete and steel are used. A secant stiffness based formulation is used for the nonlinear analysis of RC, through implicit numerical integration. The results obtained from numerical analysis using the nonlinear spring network are compared with results obtained from the experimental testing of RC beams and frame. Good agreement was observed between experimental and analytical data even when the element size was increased. Crack pattern matched the experimentally observed cracks in an averaged sense.

Once the structure has been subjected to significant damage, the analysis shifts to the EDEM phase. Analysis is performed through explicit numerical integration and simplified material models. The spring properties are adopted from the damaged springs from the first phase. The validity of the damaged spring network into the EDEM framework is checked. The complete collapse analysis of a 11-storey RC frame is performed.

Through this two-phase analysis method, a new, relatively simple method for RC frames is proposed which can, (i) model elastic behavior accurately, (ii) simulate the small deformation non-linear RC behavior, (iii) predict cracking, (iv) perform large deformation analysis, (v) model separation, collision and collapse, (vi) perform collapse analysis in a mediocre computing environment, due to less computation demand owing to the usage of large rigid circular elements, and, (vii) be used to analyze a large building stock with large material randomness, that requires a large number of analysis.

Keywords: EDEM, Spring Network, Reinforced Concrete Frame, Collapse Analysis

1. Introduction

Collapse of weak buildings has been the main cause of deaths during the past large scale earthquake disasters [1], therefore, assessing the collapse capacity of such buildings in advance is important in disaster mitigation. Assessing the collapse capacity of existing structures is not a straight forward task [2], as it involves many factors, and, it is difficult to accurately consider all these factors. This task becomes further complicated when it comes to assessing the seismic vulnerability of reinforced concrete (RC) buildings in developing countries due to additional factors like (i) lack of structural data (material properties, geometric properties, reinforcement detailing etc.), (ii) non-compliance of seismic design codes, (iii) poor construction practices, (iv) inadequate strong motion data, and, (v) socio-economic issues. In developing countries, the presence of a large number of non-engineered RC buildings, further adds to the level of uncertainty involved and the computation effort required, to obtain a reliable probabilistic vulnerability estimate.

The aim of this research is to create a numerical scheme which can be used for practical collapse simulation of RC structures with desirable characteristics like good accuracy, reliability, computationally inexpensive, simplicity in modelling and visualization. There exists various methods to assess the collapse capacity of buildings [2,3], but, there always seems to be a stand-off between the applicability/reliability of these methods and the computation effort involved. Numerical simulation of collapse gives valuable insights into the collapse mechanism of a building. Using a detailed numerical model, to model accurately the various phenomena involved in the simulation of a building collapse, is a computationally demanding process. Moreover, for better reliability, a large number of inputs of ground motions have to be considered [4], and also, to compensate for the epistemic / aleatoric uncertainties [5], a large number of simulations have to be performed, further increasing the computation effort required, sometimes, rendering these methods impractical.

There exists various analytical tools to model the transient dynamic response of a RC structure subjected to seismic loading, but they can be broadly classified based on the discretization of the material being analysed, namely the continuum based methods and the discontinuum based methods. Hybrid formulations that bridge the gap between these two kinds of methods also exist [6]. The collapse simulation of RC structures usually involves modelling of fracture of concrete, material degradation, geometric nonlinearities, instabilities, element separation, collision and collapse. The approach of discretizing the structure into an assemblage of discrete elements (discrete element methods) interacting with each other [7, 8, 9], is highly preferable due to its ability to capture element separation (crack initiation and propagation), with ease of modelling and lesser computational requirement, as compared to continuum based collapse analysis techniques [10, 11, 12, 13].

The formulation of the discrete element models had started with the Distinct Element Method (DEM) [14], which was initially introduced to model highly discontinuous and granular media. The domain was discretized into an assemblage of rigid elements with springs at the point of contact. The DEM efficiently models granular [15] and highly discontinuous media [16], and is popular in for modelling of soil rock mechanics. The DEM can also capture the behaviour of a continuous heterogeneous media like concrete [17] and rock [18].The explicit time marching scheme used in the DEM is useful to model materials under high strain loading, hence, concrete structures subjected to impact loading were analysed using the DEM [19, 20]. The aforementioned methods only consider the interaction at the surface of the elements (contact springs) which is suitable for discontinuous granular material like soil. But, in order to capture the behaviour of a continuous heterogeneous media like concrete, it is convenient to have connectivity between two adjacent discrete elements.

The Extended Discrete Element Method (EDEM) introduces a pair of joint-spring (or pore-spring), in addition to the contact springs, in order to simulate the behaviour of a continua, thus the complete material behaviour, from continuous, to fracture and finally to collapsed state can be simulated. The EDEM has previously been used to simulate concrete fracture [21], and collapse of concrete [22] and masonry structures [23].

The EDEM provides a simple and effective way to model large deformations, element separation and collapse of RC structures. In discrete analysis, a majority of the computation time is spent on contact detection [24], hence, the usage of rigid spherical elements drastically reduces the computation effort required for collapse analysis.

However, the EDEM has some limitations namely, (i) requirement of a small time step for stability and accuracy [25], due to the use of an explicit time integration scheme, (ii) no proper mechanics theory for spring stiffness derivation, and (iii) inaccuracies due to neglect of Poisson's ratio. These limitations can be overcome by the assembly of a global stiffness matrix, for the discretized system, which contains theoretically derived spring stiffness that implicitly considers the Poisson's ratio effect.

A Finite Element Mapping scheme [26] was proposed for spring stiffness derivation of a spring network discretization of an infinite elastic continuum. Through this mapping, the stiffness matrix of the assembled spring network is equal to the stiffness matrix of the Finite Element discretization it is mapped to. When an appropriate spring network is chosen, a model similar to the assembly of joint springs in the EDEM can be obtained, the difference being the effect of the contact springs i.e. inter-element interaction at the surface. By combining this property of both the aforementioned models, an effective two-phase numerical collapse simulation of structures can be performed, which can predict the initial behavior of structures (elastic/nonlinear/crack initiation/ stiffness degradation/ maximum load capacity) through the mapped spring network (implicit numerical integration) and predict the final behaviour of structures (geometric non-linearity, instability, separation, collision, and collision) through the EDEM (explicit numerical integration). To model RC, concrete and steel reinforcement springs with non-linear spatially averaged material models are used. Once the RC frame is sufficiently damaged, the analysis shifts to the conventional EDEM analysis.

In this paper, the 2-phase analysis scheme is first introduced and the simplified mapped spring network used for analysis is discussed. The material models used for modelling reinforced concrete and the analysis flow are explained. To check the applicability of the spring network to model RC frames, numerical analysis of RC beams and frames is performed and compared with experimental data. Finally the collapse simulation of an 11-storey RC framed building is performed.

2. 2-phase Analysis

The analysis of the structure comprises of two phases (Fig. 1): (i) Finite Element Mapped Spring Network Phase, (ii) Extended Distinct Element Method Phase. The overall behavior of the building from normal state to complete collapse state can be captured by these two phases.

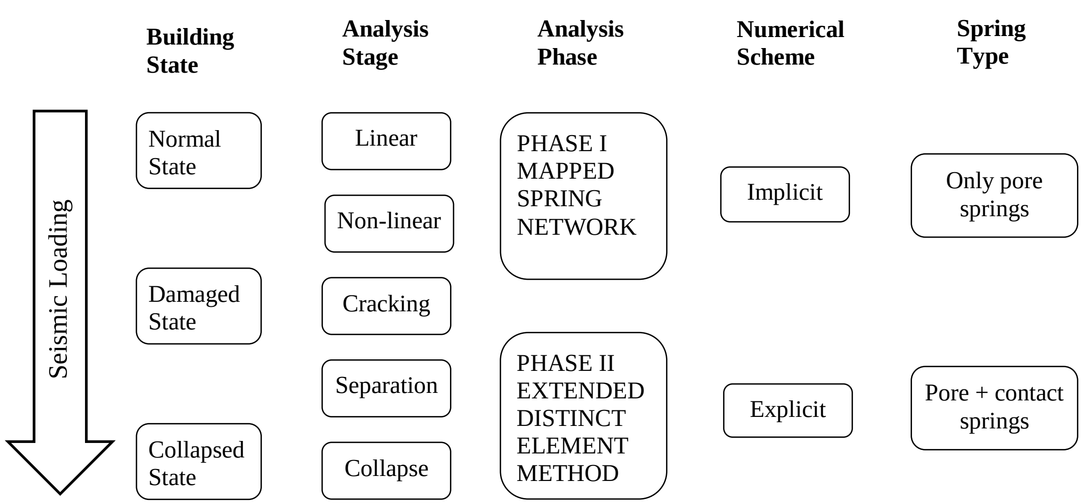


Fig. 1 – Overall Analysis Method

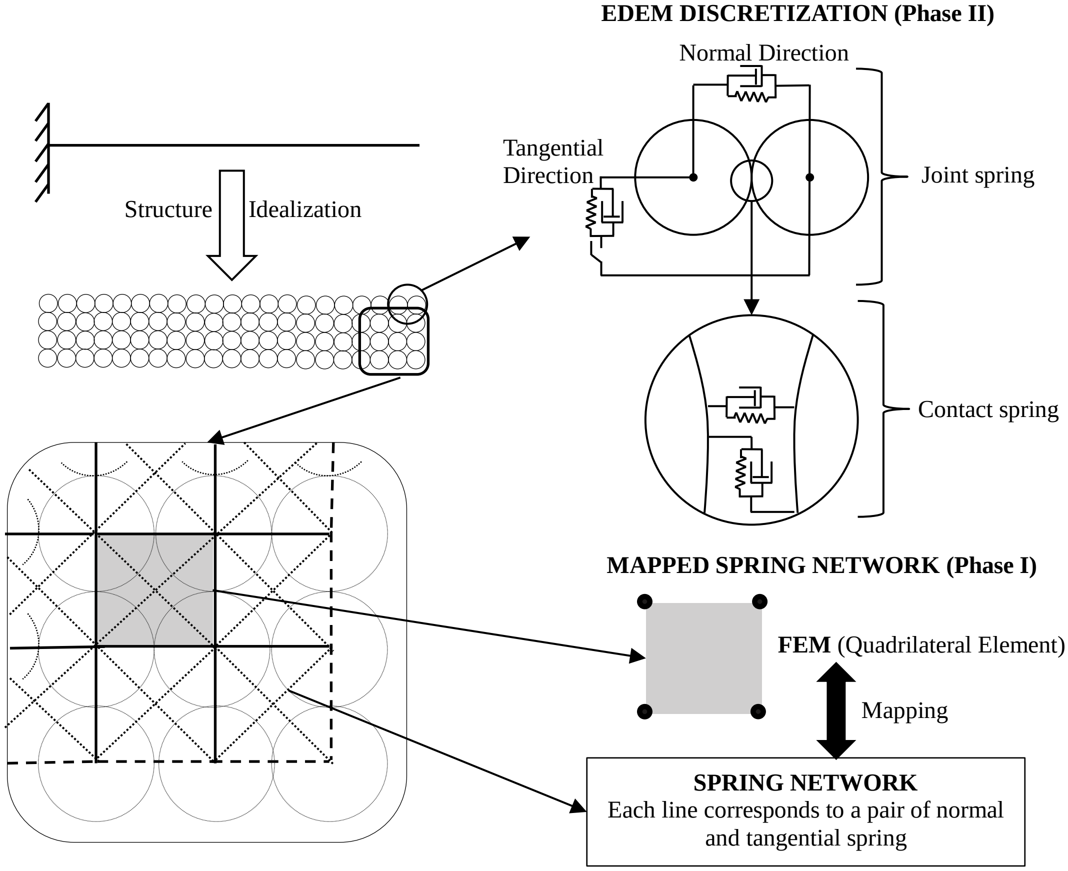


Fig. 2 – Domain discretization in EDEM and spring network

In the spring network phase, the domain is initially discretized into a spring network as shown above (Fig. 2). The spring stiffness is derived from an equivalent finite element discretization. The element shape and contact springs, do not have a meaning in this phase. The mass is assumed to be concentrated at the nodes. A sufficiently large time increment is used in dynamic analysis using an unconditionally stable, implicit numerical integration method. Once the deformations are considerable, the analysis switches to the conventional EDEM phase. The contact springs and the pore springs work at tandem, and the element shapes now have a physical meaning, for contact detection and force computation. A small time increment is used for an explicit, step-by-step time marching scheme for the dynamic analysis.

Analytical derivation of the spring stiffness of a connected DEM domain has been previously done [27, 19, 20] with the same intent of modelling fragmentation and collapse, using a grillage of structural elements, consisting of cylindrical elements in a closed hexagonal packing. However, its applicability for complex domains and large deformation analysis is not yet verified and also, there is a restriction to the kind of basic unit of discretization used. Simple lattice models, comprising of an assemblage of 1-D truss elements have been used along with EDEM to simulate collapse of RC bridges [28]. However, it has been observed that shear springs have to be incorporated into the lattice spring model, in order to allow the full range of the Poisson’s ratio of the solid to be modelled [29].

Table 1 – Spring stiffness and local spring orientation for CST elements

Spring Type		Spring stiffness (Eigen Value) (Plane Strain)	Spring direction (Eigen direction)	Spring stiffness (when ν= 1/3) (Plane Stress)
Inner	normal	$\frac{3\lambda+5\mu}{2\sqrt{3}}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (Orthogonal)	$\frac{\sqrt{3}\,E}{2}$
	tangential	$\frac{-\lambda+\mu}{2\sqrt{3}}$		0
Boundary	normal	$\frac{\lambda+3\mu+A}{4\sqrt{3}}$	$\begin{pmatrix}\frac{2\lambda+2\mu+A}{\sqrt{3}(\lambda-\mu)} & \frac{2\lambda+2\mu-A}{\sqrt{3}(\lambda-\mu)}\\ 1 & 1\end{pmatrix}$ where, $A=\sqrt{\lambda^2+14\lambda\mu+\mu^2}$	$\frac{\sqrt{3}\,E}{4}$
	tangential	$\frac{\lambda+3\mu-A}{4\sqrt{3}}$		0

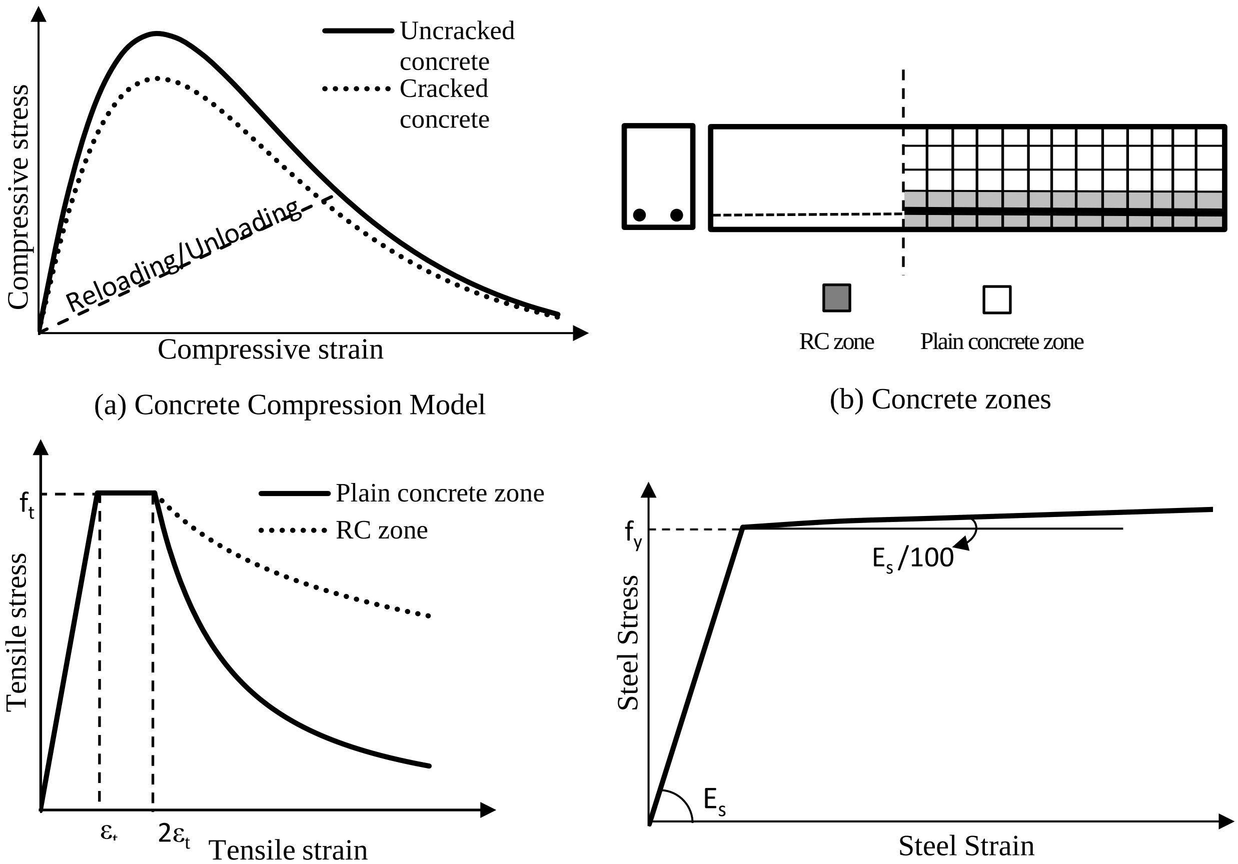
λ,μ *Lamé constants* E, *Elastic Modulus*

3. Finite Element Mapped Spring Network

The Finite Element mapping scheme for spring network representation of mechanics of solids [26], presented a rigorous method for spring stiffness derivation for any anisotropic infinite media. The spring stiffness of the spring network is derived by invoking the property of translation invariance of the stiffness matrix. The spring stiffness is found by equating the off-diagonal blocks of both the global stiffness matrices, derived by finite element discretization, and, from assembly of a spring network respectively. The spring stiffness can also be obtained by finding the eigenvalues of the off-diagonal block it corresponds to. The eigenvalues correspond to the spring stiffness and the eigenvectors correspond to the orientation of the springs.

For the current simulation, the simplest element used in finite element discretization is the constant strain triangle (CST) element is considered. It was observed that there were two kinds of springs for the triangular spring network discretization namely, the inner spring and the boundary spring (Table 1). The inner springs, are mutually orthogonal to each other. A special case, when λ=μ, then the tangential stiffness becomes zero and the spring network for an infinite domain (no boundary springs), becomes a network of one directional Hook's law springs, with spring stiffness $k=\frac{4}{\sqrt{3}}\lambda=\frac{4}{\sqrt{3}}\mu$ [30]. In the case of plane stress formulation, when Poisson's ratio is 1/3 it leads to a simplified 1-D spring system.

The spring network obtained from the finite element mapping using the aforesaid elements, has only, one central two-body interaction (normal spring) and one non-central two-body interaction (tangential spring), limiting the rotational degree of freedom at the nodes. This spring network is similar to the Born Lattice Model [31, 32], and is a simple and effective model, but, this model is not rotationally invariant [33]. This problem can be solved by (a) considering rotational degree of freedom in parent finite element (beam element) (b) using a local strain based deformation measurement [29]. But, since the finite elements stiffness are derived based on infinitesimal strain theory, and, for small deformations, the effect of this invariance is unimportant, and hence can be neglected [34]. The change in length of spring is used during computation to detect rigid body rotation, and if excessive rigid body movement is present (due to yielding/excessive cracking), the analysis shifts to the EDEM phase, which considers the elemental rotation in its computation. For simplicity, currently the 1-D CST spring network is considered, however, shear spring can be incorporated appropriately.



(a) Concrete Compression Model

(b) Concrete zones

(c) Concrete Tension Model

(d) Reinforcement material model

Fig. 3 – Reinforced concrete material models

4. Non-linear Reinforced Concrete Analysis

A smeared crack approach for RC, consisting of spatially averaged constitutive models for concrete and reinforcing steel [35], is adopted for the springs. Finite element analysis with these material models have shown to accurately model the nonlinear response of RC. The spring network model is analogous to the fixed smeared crack approach with the active crack assumed to be propagating along a pre-defined direction. The failure criterion is the stresses computed along the direction of the springs which puts a restriction on the direction of the propagating smeared crack; the crack is assumed to be propagating perpendicular to the cracked springs.

A compressive material model (Fig. 3(a)) based on the Elasto-Plasto Fracture (EPF) model [36] is used. The advantage of using these models is that they are based on uniaxial constitutive relations, using parameters which are obtained from simple experiments, for example the uniaxial compressive strength (f_c). The cracking of springs is initiated when the tensile springs reach the maximum tensile stress.

The elastic modulus of the tension model before cracking reduces based on a fracture parameter (K_0). Cracking criteria is a function of the tensile strength (f_t) and K_0. Post cracking the tensile spring exhibits softening (Fig. 3(b)) and the slope of the softening curve depends on the element size, and also, the concrete zone it exists (Fig. 3(b)). In the RC zone, the slope of the softening curve is small, which represents the tension stiffening effect of the concrete in the vicinity of the reinforcement. For concrete springs in Plain Concrete (PC) zone the slope of the descending curve varies depending on the size of the element, in order to maintain constant fracture energy released per unit area, which ensures mesh objectivity of results.

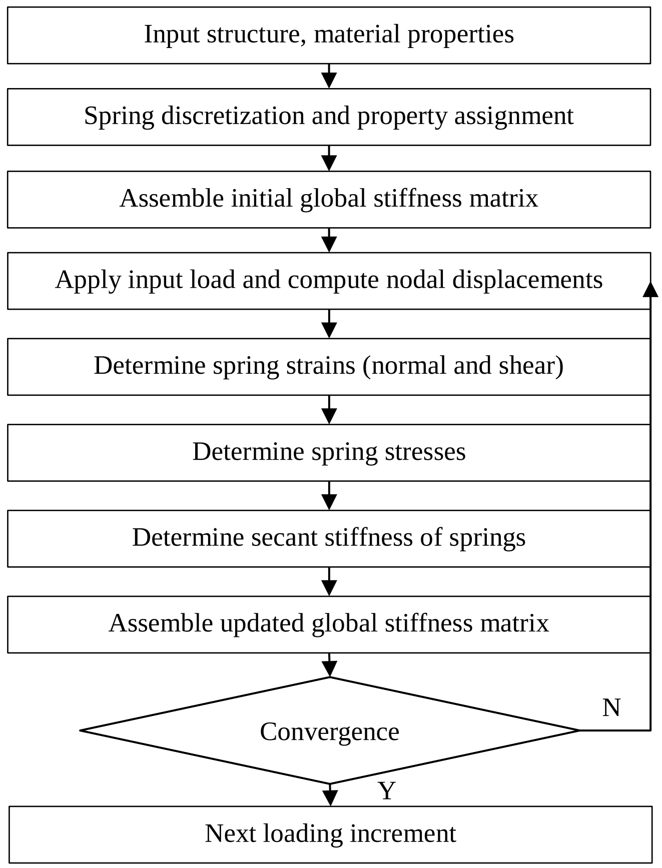


Fig. 4 – Flow Chart of the analysis procedure

A simple bi-linear stress-strain model (Fig. 3(d)) is used to model the reinforcement. A very small modulus of elasticity (1/100) is assumed after yield. Currently, the shear resistance of reinforcement bars (dowel action) and the bond-slip of reinforcement bars are not considered.

An iterative secant stiffness based formulation is used for computation, as it is numerically stable while handling complex material models [37] and post peak response of structures. At every iterative step, secant modulus value of the total stress-strain relationship of the springs is updated till convergence. The flow of the program can be found above (Fig. 4). One of the advantages of this method is that, good accuracy can be obtained even with the use of relative simple finite elements.

5. Experimental Validation

To check the validity of the spring network, numerical simulation of experimentally tested structural elements are performed. The validity of the simple CST spring network to model simple RC beam and RC frame are checked.

5.1 RC Beam

A series of testing was performed on 12 beam specimens with varying length, breadth and reinforcing ratio [38]. The length and breadth of the specimens were varied accordingly in order to induce various kinds of damage modes. The spring network discretization of the beams can be below (Fig. 5). The specimen is subjected to central loading and the mid span deflection is measured. The load-deflection curves obtained during experimentation are compared with the results obtained through numerical analysis (Fig. 7). The crack springs obtained from the analysis (Fig. 6) correspond to the crack pattern obtained from experimentation. The comparison of results show the capability of the CST spring network to predict the non-linear behaviour of RC beams using simple 1D material models.

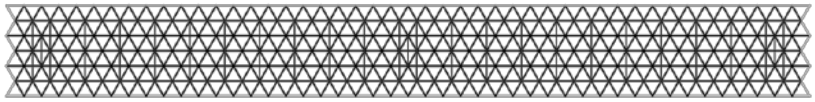


Fig. 5 – Spring network model of the Beam

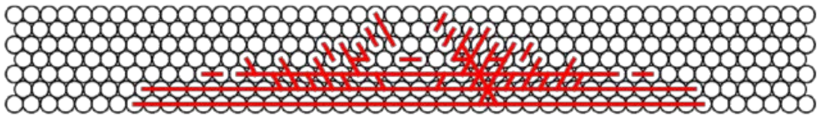


Fig. 6 – Failed Springs

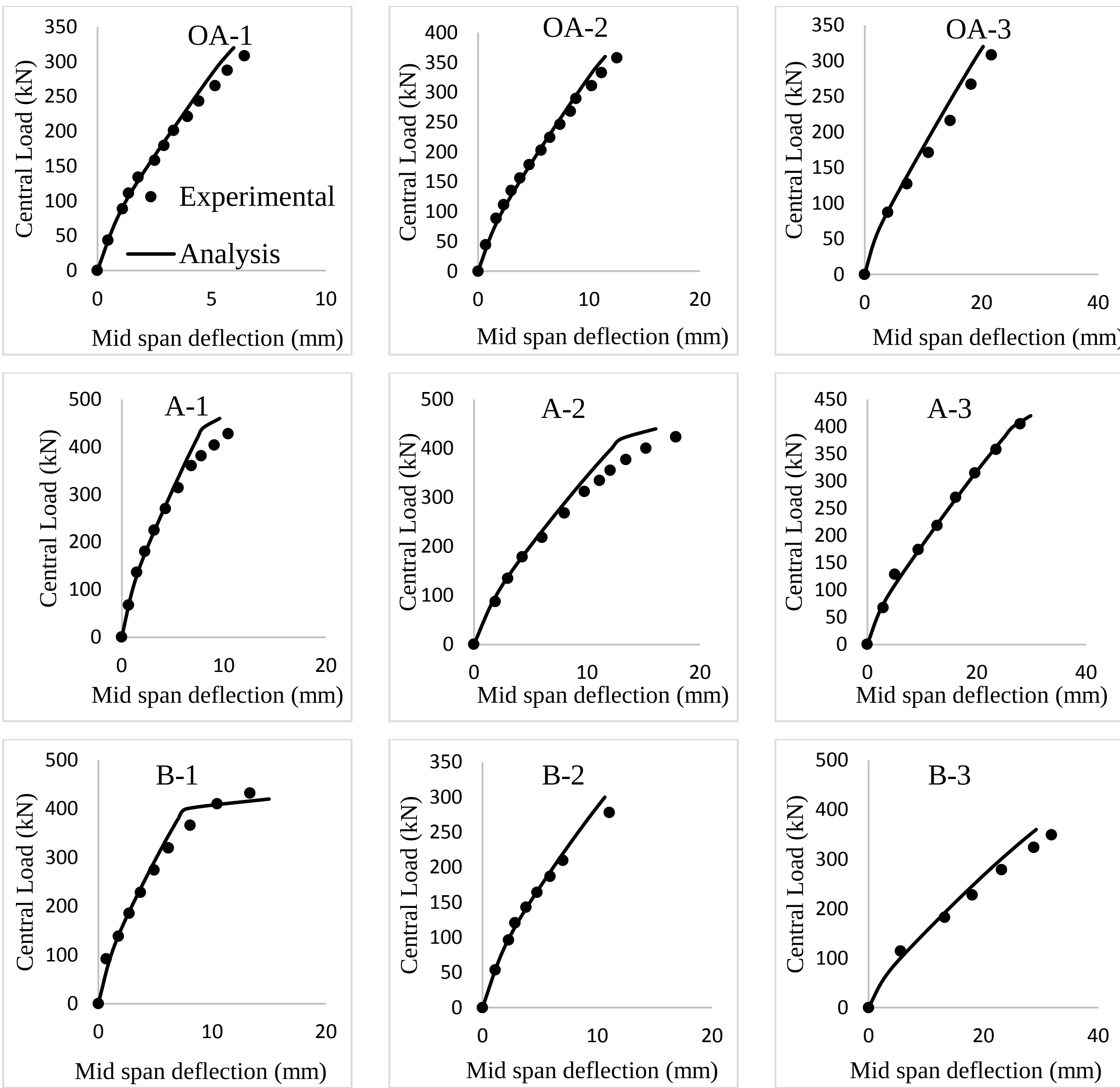


Fig. 7 – Comparison between experimental and simulated force-deformation response of the beam

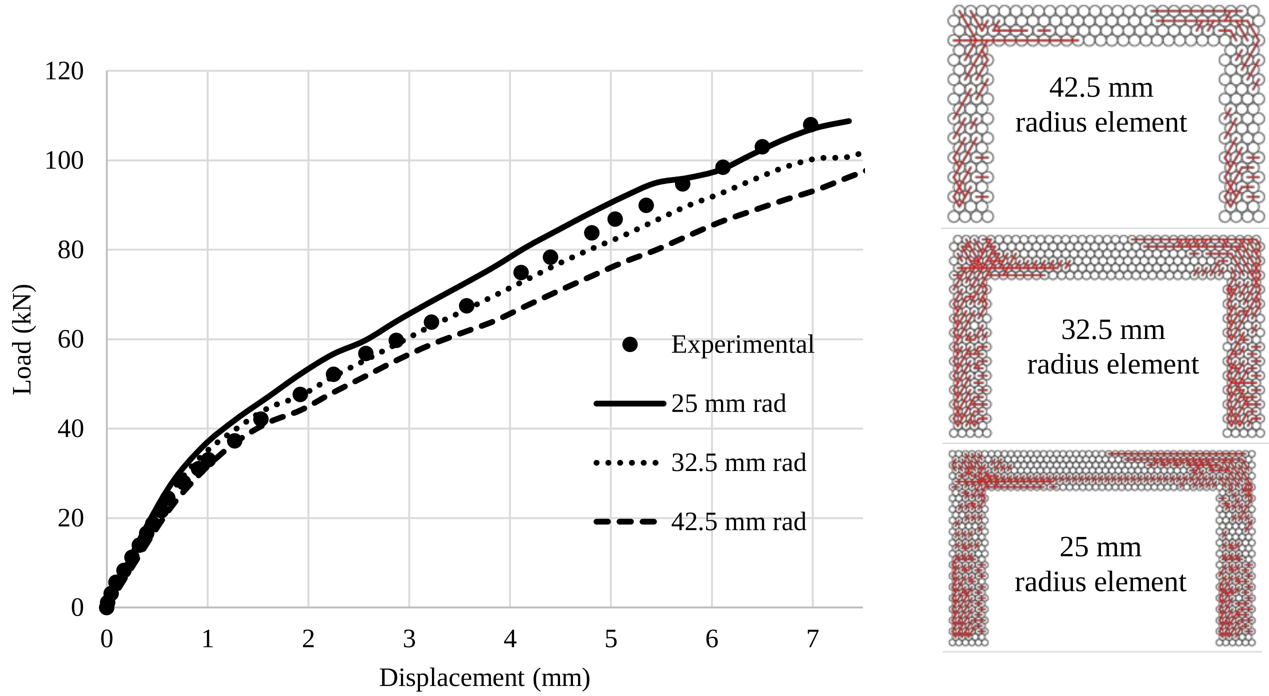


Fig. 8 – Comparison between experimental and simulated response of the RC frame for various element sizes and the pattern of cracked springs

5.2 RC Frame

The results obtained from the lateral loading experimentation on a RC frame [39] are used to check the capability of the spring network. The analysis is performed with three kinds of discretization with varying element sizes radius viz. 25 mm, 32.5 mm and 42.5 mm (Fig. 8). The lateral load-displacement curve obtained through numerical analysis can be seen above. Not much variation was observed even when larger element sizes were used. The element size is dictated by the assumption made of the size of fracture process zone by the smeared crack approach. The results show the capability of the simplified spring network to predict the nonlinear behaviour of a RC frame.

6. Collapse Simulation

Once the frame is sufficiently damaged the analysis shifts to the EDEM phase. The circular element shapes play an important role in this phase, as they are used to compute the contact forces. The springs follow simplified linear material models with Mohr-Coulomb rupture criteria. The spring stiffness is reduced based on the secant modulus reduction due to the damage incurred from the first phase. Springs that have failed in tension are cut off.

An 11-storey experimentally tested (1/15 scale) RC frame [40] is simulated using the EDEM. First, the stability of the damaged frame is checked under gravity loading and then the frame is subjected to further loading till the frame collapses (Fig. 9). The EDEM can effectively follow the large deformation and collapse of the frame.

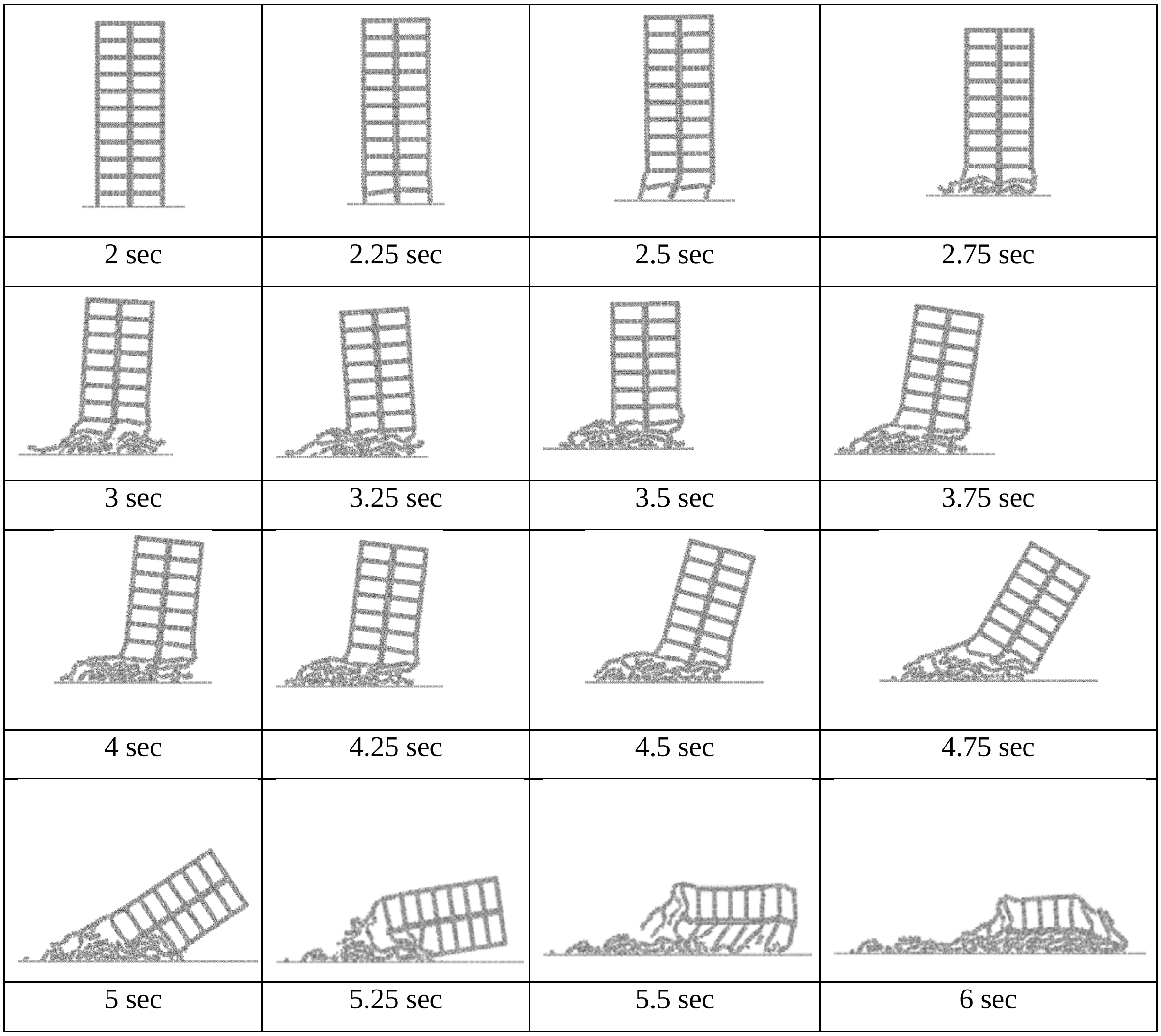


Fig. 9 – Collapse simulation of a 11-storey frame

7. Conclusion

A numerical model has been developed for reinforced concrete frames which can (i) model elastic behaviour accurately (ii) model RC non-linearity, cracking, stiffness degradation, and can also predict ultimate load (iii) simulate large deformation, separation, collision and complete collapse (iv) reduce the overall computational effort required due to the large, circular and rigid elements being used (v) be used to model a large number of building stock, especially in developing countries (vi) be used to perform a large number of simulation, to obtain a reliable vulnerability assessment of an RC framed building, considering material and geometric randomness of the RC frame.

This method is suitable to simulate the collapse of simple domains such as RC Frames. Due to the simplifications made due to convenience and reducing the computing cost, this method cannot be applied to complicated concrete structures with a complex state of stress. In usual nonlinear finite element concrete analysis, nonlinearity is induced at the points of integration, finding the cracking plane through appropriate stress transformations. But in this case, nonlinearity is induced through the springs in pre-determined directions (along the length of the spring), which further restricts its application. However, within the scope of simulating RC frames, this method has many advantages in the practical vulnerability assessment of RC framed structures.

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