

A CRITICAL REVIEW OF THE RAYLEIGH DAMPING MODEL

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Abstract

This paper summarizes a study aimed at evaluating damping ratios in buildings inferred from acceleration records obtained in instrumented buildings in California. The data was obtained by examining 119 seismic responses, coming from 24 buildings, collected over the past 25 years by the California Strong Motion Instrumentation Program. All the records were analyzed using a least-squares system identification technique in the time domain. Using only reliable data, the variation of damping with modal frequencies is examined for all buildings. It is found that in 96% of the cases, the modal damping ratios increase approximately linearly with frequency, showing that damping in buildings is best represented by using a stiffness proportional damping. No evidence was found to suggest that a mass-proportional model could be appropriate. The distribution of the linear functions' parameters are then studied and compared to other published damping recommendations. The results of this investigation are of paramount importance as they could lead to a drastic departure in how damping has been modeled in buildings to date.

Keywords: damping, system identification, higher modes, Rayleigh damping



1. Introduction

The estimation of the response of buildings to seismic motion requires knowledge of the damping ratio of the different modes of vibration that contribute to it. Certain response quantities, like peak floor accelerations (PFA), peak floor velocities (PFV), and peak interstory drift ratios (PIDR) typically have a very significant contribution of higher modes. These response quantities are usually used as demand parameters for structural and non-structural components. For example, PFA, are an important demand parameter for acceleration-sensitive components, such as elevators, air-conditioning systems, emergency power generator systems, etc.; PFV is commonly used to compute the response of components that are not rigidly attached to the floor, and whose response is dominated by sliding; and it has been shown that PRDR is the demand parameter that is best correlated with damage in buildings [1 - 4]. Consequently, an accurate estimation of these important engineering demand parameters requires an appropriate modeling of damping in higher modes.

Originally proposed by John W. Strutt (aka Lord Rayleigh) in 1877, the so-called Rayleigh damping is the most common model used to incorporate damping in structures for assessing their seismic response. Lord Rayleigh [5] found that the system of equations could be uncoupled if what he referred to as "dissipative forces" F were assumed to be proportional to either the kinetic energy of the system T, or to the potential energy of the system V. He also noted that "the same exceptional reduction is possible when F is a linear function of T and V, or when T is itself of the same form as V". When using this damping model in a matrix formulation, the damping matrix is assumed to be a linear combination of the mass and stiffness matrices with coefficients selected to obtain a desirable damping ratio at two modal periods/frequencies. Although this damping model leads to a convenient simplification that allows the uncoupling of modal equations in the case of linear response, the validity of the model has received limited attention. Far more discussion has been devoted to the use of the model in nonlinear structures, and in the case of the stiffness-proportional portion of the damping matrix, whether it should be proportional to the initial, secant or to the tangent stiffness; but little attention has been devoted to its validity even for the case of structures responding elastically. This lack of attention is, in part, due to the scarcity of high quality damping data for higher modes.

There is only a handful of studies that investigate damping in higher modes. In 1972, Yokoo and Akiyama [6] collected damping data from the vibration tests of 17 steel buildings. They showed that the damping of higher modes increased with frequency, but that this relationship did not correspond to a stiffness proportional model. Three years later, Hart and Vasudevan [7] studied the records of 12 buildings subjected to the San Fernando Earthquake of 1971. They estimated the damping ratios of the first three modes using a simplified identification technique. They did not find a significant difference between the damping ratio of the fundamental mode and that of higher modes. Therefore, for earthquake-resistant design, they recommended using a constant damping value for all modes. In 1976, O'Rourke [8] analyzed the damping data available in the literature for buildings under wind motions. He showed that in 61% of the cases, the damping of the second mode was higher than in the first mode. The same occurred between the third and second modes in 53% of the cases. Based on this study, Kareem [9] proposed a formula for modal damping based on the frequency ratio with respect to the fundamental mode. This equation followed a stiffness proportional model. Kareem and Gurley [10] examined the applicability of the latter predictor by testing it against new data from Tamura et al. [11] and other sources, concluding that it was a satisfactory representation of the observed trend. Satake and other investigators [12] studied the damping ratios of 205 buildings subjected mostly to low-amplitude motions such as ambient vibrations or forced vibration tests. They examined the first three translational modes, finding that damping increases with frequency at a rate lower than with stiffness proportional damping.

The Center for Engineering Strong Motion Data (CESMD) is a cooperative center that integrates the earthquake strong-motion data collected by the US Geological Survey (USGS) and the California Geological Survey (CGS). Their database of monitored structures includes buildings that have been subjected to multiple ground motions. This provides an opportunity to make numerous observations of the damping ratios in these buildings when subjected to earthquakes. The purpose of this work is to determine whether the Rayleigh model – widespread as it is – has empirical validity. To accomplish this, the relationship between modal damping ratios and their corresponding frequencies is examined for 24 buildings subjected to earthquake motions. All the



damping ratios were estimated using a parametric system identification technique based on the minimization of the error between the recorded and predicted responses in the time domain [13]. Each data point was carefully examined, passing a series of reliability tests to ensure that only high quality damping ratios were considered. The obtained results suggest that a new approach for modeling damping is necessary if we want to correctly capture the observed data.

2. Rayleigh Damping

Viscous damping is a simple mathematical representation of the energy dissipation in a structure. The damping matrix groups the contribution of all the sources of energy dissipation while the structure remains in its linear-elastic range. Given the wide variety of energy-dissipating mechanisms, the damping matrix cannot be assembled from the properties of the individual components of the structure, like the mass or stiffness matrices. Instead, the damping matrix is assembled from specifying the individual modal damping ratios. The Rayleigh damping model considers the damping matrix [C] to be proportional to the stiffness and mass matrices, [K] and [M] respectively. That is:

$$[C] = a_0[M] + a_1[K] \tag{1}$$

where a_0 and a_1 are the proportionality constants, with units of s⁻¹ and s, respectively. The damping ratio of the k-th mode is then given by:

$$\xi_k = \frac{1}{2\omega_k} a_0 + \frac{\omega_k}{2} a_1 \tag{2}$$

where ω_k corresponds to the circular frequency of the k-th mode. The constants a_0 and a_1 are obtained by specifying the damping ratios of two different modes and using Eq. (2). Fig. 1 shows the variation of damping with frequency. A stiffness-proportional model is defined by the second term of Eq. (2), obtaining a linear variation with frequency. Stiffness proportional damping can be interpreted as a model of the structure having dashpots between consecutive stories (Fig. 2, right). In this model, energy dissipation can be thought of as being the result of interstory motion, and the distribution of damping forces will be proportional to the relative velocities between floors. In a mass-proportional model, damping is defined by the first term of Eq. (2), obtaining a hyperbolic decrease of damping with frequency. This is equivalent to connecting a dashpot from each story to a fixed base (Fig. 2, left). In this case, the damping forces are proportional to each story's relative velocity to the ground, causing a load profile that increases along the height of the building. Such a load profile is not feasible for an actual building because it would require the structure to be immersed in some sort of



viscous fluid.



Fig. 1 – Variation with frequency of the mass-proportional, stiffness proportional, and Rayleigh damping models



Fig. 2 - Mass-proportional (left), and stiffness-proportional (right) damping models

3. System Identification

In a linear, modal superposition model, the response of each mode is governed by the following equation of motion:

$$\ddot{D}_n(t) + 2\omega_n \xi_n \dot{D}_n(t) + \omega_n^2 D_n(t) = -\ddot{u}_g(t)$$
⁽³⁾

where D_n , D_n , and D_n correspond to the relative acceleration, velocity, and displacement of a single degree of freedom system with unit mass, natural circular frequency ω_n , and critical damping ratio ξ_n ; \ddot{u}_g is the ground acceleration, and the sub index *n* denotes the mode number. The modal displacements u_{nj} at the j-th degree of freedom of the structure can then be computed as:

$$u_{ni}(t) = \Gamma_n \phi_{ni} D_n(t) \tag{4}$$

where Γ_n is the modal participation factor, and ϕ_{nj} is the mode shape for mode *n* evaluated at the *j*-th degree of freedom. Multiplying Eq. (3) by $\Gamma_n \phi_{nj}$ yields:

$$\ddot{u}_{nj}(t) + 2\omega_n \xi_n \dot{u}_{nj}(t) + \omega_n^2 u_{nj}(t) = -\Gamma_n \phi_{nj} \ddot{u}_g(t)$$
⁽⁵⁾

where \ddot{u}_{nj} , \dot{u}_{nj} , and u_{nj} correspond to the acceleration, velocity, and displacement of the *j*-th degree of freedom for mode *n* relative to the ground, respectively.

If only *N* modes are considered to have a significant influence in the seismic response of the building, then the response of the structure can be calculated as:

$$\ddot{\hat{u}}_{j}^{t}(t) \approx \ddot{u}_{g}(t) + \sum_{n=1}^{N} \Gamma_{n} \phi_{nj} \ddot{D}_{n}(t)$$
(6)

where $\ddot{\hat{u}}_{j}^{t}$ is the predicted total (absolute) acceleration at the j-th degree of freedom.



The objective function J considered in this study is defined as the difference squared between the predicted relative acceleration \ddot{u} and the one measured by the sensors in the building \ddot{u} . This difference is then normalized by the measured relative acceleration summed over all times and sensors:

$$J(\Theta) = \sum_{j=1}^{N_{sen}} \sum_{i=1}^{\tau} \frac{\left[\ddot{u}_j(i\Delta t) - \ddot{\tilde{u}}_j(i\Delta t) \right]^2}{\sum_{k=1}^{\tau} \left[\ddot{u}_j(k\Delta t) \right]^2}$$
(7)

Where N_{sen} is the number of sensors above ground level; Δt and τ are the time step and the number of points in the signal, respectively. The purpose of the normalization is to provide an equal weight to each sensor location. Please note that if this normalization is not done, the identification would converge towards parameters that produce a better fit in floors experiencing larger accelerations.

The optimal set of parameters will be the one that minimizes the objective function:

$$\min J(\Theta) \tag{8}$$

where, if the structure is assumed to be at rest at t = 0:

$$\Theta = \left(\begin{cases} \omega_1 \\ \vdots \\ \omega_N \end{cases}; \begin{cases} \xi_1 \\ \vdots \\ \xi_N \end{cases}; \begin{bmatrix} \Gamma_1 \phi_{N1} & \cdots & \Gamma_N \phi_{N1} \\ \vdots & \ddots & \vdots \\ \Gamma_1 \phi_{1N_{sen}} & \cdots & \Gamma_N \phi_{NN_{sen}} \end{bmatrix} \right)$$
(9)

Details on the optimization problem, such as boundary and initial conditions, mode inclusion criteria, and reliability tests for the identified damping ratios can be found in reference [14].

4. Buildings Analyzed

A total of 119 seismic responses, coming from 24 buildings in California, were analyzed. Table 1 shows a summary of the studied buildings, including their location, number of stories, height, lateral force resisting system, material, and number of analyzed records. All the data was obtained from the database of the California Strong Motion Instrumentation Program (CSMIP). For each recorded earthquake, the buildings in the data set were analyzed in two perpendicular directions: north-south (NS) and east-west (EW). Schematics of the buildings' plans, elevations, and distribution of sensors can be obtained through the CSMIP website [15].

5. Identified Damping Ratios

Fig. 3 and Fig. 4 show the relation between the identified damping ratios and natural frequencies for the 48 cases analyzed. Each plot also shows a linear regression of the data with its respective \mathbb{R}^2 value. An overall increase in damping with frequency is observed in 46 of the 48 cases (96%). It can be seen that this increment is roughly linear, suggesting that damping might be best represented by a stiffness-proportional model. The linear trend is better appreciated in tall, flexible buildings with identified values up to the 5th mode, like LA-52, LA-54, and LA-32; but can also be clearly appreciated in lower, more rigid buildings like LA-09, or buildings with concrete shear walls like LA-14. A pure stiffness-proportional model, however, would require the intercept of the linear trend to be zero. This is not observed in any of the buildings. None of the buildings show evidence to suggest that a mass-proportional model could be appropriate either. The linear trends suggest that a better approach is to model damping ratio as consisting of two parts: a frequency independent term ξ_0 and a frequency proportional term:



$$\xi(f) = \xi_0 + \beta \cdot f \tag{10}$$

For each building, ξ_0 and β can be obtained from the intercepts and slopes of the linear regressions, respectively. Assuming a lognormal distribution for both parameters, estimates of the logarithmic mean $\mu_{\ln\xi}$ and standard deviation $\sigma_{\ln\xi}$ were calculated from the sample mean and the unbiased estimate of the log of the positive data. Fig. 5 shows the empirical cumulative distribution functions (CDF) of the slopes and intercepts obtained from the regression data of all buildings, as well as their corresponding fitted lognormal distributions.

#	ID	CSMIP ID	Location	Stories	Height (m)	Str. Sys*	Material**	# Records
1	RE-07	23481	Redlands , CA	7	28.8	MF	S	5
2	RI-07	13702	Riverside , CA	7	49.1	BF	S	5
3	IR-08	13329	Irvine , CA	8	35.7	SW+MF	S+RC	3
4	CO-09	58638	Concord, CA	9	38.9	MF+BF	S	4
5	LA-09	24248	Los Angeles , CA	9	44.8	BF	S	11
6	PA-09	24571	Pasadena, CA	9	41.5	MF	RC	6
7	RI-10	13415	Riverside, CA	10	42.7	MF	S	2
8	WC-10	58364	Walnut Creek, CA	10	39.2	SW	RC	8
9	OA-11	58337	Oakland , CA	11	43.6	SW	RC	7
10	PA-12	24566	Pasadena, CA	12	51.2	MF	S	6
11	SO-13	24322	Sherman Oaks , CA	13	50.0	MF	RC	3
12	ES-14	14654	El Segundo , CA	14	61.9	MF+BF	S	6
13	LA-14	24236	Los Angeles , CA	14	45.3	SW	RC	9
14	LB-15	14533	Long Beach , CA	15	80.8	MF	S	3
15	RC-16	58615	Redwood City, CA	16	67.8	MF+BF	S	3
16	LA-19	24643	Los Angeles , CA	19	86.3	BF	S	1
17	SD-19	3603	San Diego , CA	19	80.8	BF	S+RC	4
18	LA-20	24464	North Hollywood, CA	20	51.5	MF	RC	4
19	OA-22	58312	Oakland , CA	22	100.0	MF	S	2
20	SD-21	3300	San Diego , CA	21	92.2	MF	S	4
21	LA-32	24288	Los Angeles , CA	32	102.7	MF	S	8
22	SF-47	58532	San Francisco , CA	47	171.9	MF	S	1
23	LA-52	24602	Los Angeles , CA	52	218.2	BF+OR	S	7
24	LA-54	24629	Los Angeles , CA	54	218.1	MF	S	7

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(*) MF: Moment frames, BF: Braced frames, SW: Shear walls, OR: Outriggers

(**) RC: Reinforced concrete, S: Steel

In stiffness-proportional damping, energy dissipation can be thought of as being proportional to the interstory motions. The mode shapes of higher modes induce larger interstory deformations in the building, so damping is expected to increase with the mode number, hence, with frequency. For the same reason, the rate of damping increment is expected to be higher in flexible buildings than in rigid buildings. Fig. 6 shows the empirical CDF of the slopes and intercepts, respectively, of all the steel buildings in the dataset, separated by a lateral resistant system. It can be seen that the moment frame buildings have, on average, a higher slope and intercept than braced frame buildings. This can be explained by the increased flexibility of one structural system over the other. The frequency independent term ξ_0 varies largely from building to building because, as shown in Cruz and Miranda [14], the damping ratio of the fundamental period depends on the height of the building.



Fig. 3 – Identified damping ratios versus frequency



Fig. 4 - Identified damping ratios versus frequency



Fig. 5 – Cumulative distribution of the frequency proportional term β (left) and frequency independent term ξ_0 (right), and corresponding fitted lognormal distributions



Fig. 6 – Cumulative distribution of β (left) and ξ_0 (right), for steel moment frames, and steel braced frames and mixed structural systems data; and their corresponding lognormal distribution fitted curves

6. Comparison with Other Damping Recommendations

This section compares the observed data, and corresponding regressions, with other damping models available in the literature. First, the evolution of damping with frequency is analyzed and compared to the model proposed by Kareem and Gurley [10]. Then, the evolution of damping with mode number is studied and compared to the results obtained by Satake et al. [12].

Kareem and Gurley [10] proposed a stiffness-proportional formula to compute the normalized damping ratio of the *n*-th ξ_n / ξ_1 , as a linear function of the normalized natural frequencies f_n / f_1 :

$$\frac{\xi_n}{\xi_1} = 1 + C \cdot \left(\frac{f_n}{f_1} - 1\right) \tag{11}$$



Using the data from Yokoo and Akiyama [6], they determined the constant C to be approximately equal to 0.38. In the current investigation, for each analyzed building, an estimate $\tilde{\xi}_1$ of the damping ratio of the first mode was computed by evaluating the linear regressions shown in Fig. 3 and Fig. 4 at their corresponding average inferred fundamental frequency \tilde{f}_1 . Fig. 7 shows the variation of damping with frequency for all buildings, normalized by $\tilde{\xi}_1$ and \tilde{f}_1 , respectively. A linear regression of the data suggests that the value of C should be corrected to 0.12. The figure compares the obtained linear regression with the values proposed by Kareem and Gurley [10]. It can be seen that this recommendation significantly overestimates the inferred damping ratios. Fig. 7 also shows the prediction of a Rayleigh damping model. This model was calculated with Eq. (2) assuming the same damping ratio for two specified frequencies: f_1 and $5f_1$, a common assumption in the engineering practice. It can be seen that the Rayleigh damping model consistently underestimates damping at all the modal frequencies that contribute to the structural response. The mass and stiffness proportional damping models always provide lower damping values than the Rayleigh model (Fig. 1). Therefore none of these models, by themselves, are good approximations of the observed trend.



Fig. 7 - Normalized damping and comparison of different damping models

Satake et al. [12] studied the variation of damping ratios between two adjacent modes, proposing the following expression:

$$\xi_n = \alpha \xi_{n-1} \tag{12}$$

They recommended α values based on the building's material: between 1.3 and 1.4 for steel framed buildings, 1.4 for reinforced concrete buildings, and between 1.7 and 1.8 for combined steel and concrete buildings. These values, however, may only be applied to the lower modes (second and third) of high-rise buildings. Table 2 shows the mean and standard deviation of the α values calculated in the current investigation. They consider the identified damping ratios of all the buildings in the dataset. For the second and third modes, the mean α values are 1.46 and 1.29, respectively, which is very close to the approximate values recommended by Satake et al. [12]. Nevertheless, this approach is not recommended because it does not take into account the frequency dependency of damping ratios. A better approach is to analyze the evolution of damping with the mode number n, as shown in Fig. 8. It can be seen that damping clearly increases with mode number; a linear regression of the data yields:

$$\frac{\xi_n}{\xi_1} = 0.27n + 0.77 \tag{13}$$



Mode number n

5



Table 2 – Statistics of the quotient between damping ratios of two consecutive modes

3

4

2

Fig. 8 - Variation of normalized damping with mode number

7. Conclusions

The validity of the mass-proportional and stiffness-proportional assumptions in Rayleigh damping was examined by studying damping ratios inferred from the analysis of 119 seismic responses, coming from 24 buildings in California. It was shown that modal damping ratios increased with frequency in 96% of the analyzed cases, and that the rate of increment was approximately linear. It was concluded that damping is best represented by the sum of a frequency independent term and stiffness-proportional term. These parameters were shown to be approximately distributed as lognormal random variables, with logarithmic means of 1.04 and -0.15, and logarithmic standard deviations of 0.56 and 0.75, respectively. No evidence was found to suggest that a mass-proportional model should be appropriate. It was found that the Rayleigh damping model, when applied using typical engineering practice assumptions, leads to an underestimation of damping of higher modes that contribute to the seismic response therefore also leading to an overestimation of the seismic response.

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9. References

- [1] Pavlou, E., and MC Constantinou (2006): Response of nonstructural components in structures with damping systems. *Journal of Structural Engineering* **132** (7):1108–1117.
- [2] Algan, B (1982): Drift and damage considerations in earthquake-resistant design of reinforced concrete buildings. PhD. Dissertation, University of Illinois at Urbana-Champaign.
- [3] Moehle, BJP (1984): Strong motion drift estimates for R/C Structures. *Journal of Structural Engineering* **110** (9), 1988–2001.
- [4] Bozorgnia Y, and V. Bertero (2001): Improved shaking and damage parameters for post-earthquake applications. Proceedings, *Proceedings SMIP01 seminar on utilization of strong-motion data*, Los Angeles, CA, 1–22.
- [5] Strutt, J. (1877): The theory of sound. Vol. 1, 2nd ed., London and New York, Macmillan and Co.
- [6] Yokoo, Y., and H. Akiyama (1972). Lateral vibration and damping due to wind and earthquake effects. *Proceedings of the International Conference On Planning and Design of Tall Buildings*, New York, NY, 619-634
- [7] Hart, G., and R. Vasudevan (1975): Earthquake design of buildings: damping. *Journal of the Structural Division* **101** (1), 11-30.
- [8] O'Rourke, M. (1976): Discussion of Response to Stochastic Wind of N-Degree Tall Buildings. *Journal of the Structural Division* **102** (12), 2401-2403.
- [9] Kareem, A. (1981): Wind-excited response of buildings in higher modes. *Journal of the Structural Division* **107** (4), 701-706.
- [10] Kareem, A., and K. Gurley (1996): Damping in structures: its evaluation and treatment of uncertainty. *Journal of Wind Engineering and Industrial Aerodynamics* **59** (2), 131-157.
- [11] Tamura, Y., M. Yamada, and H. Yokota (1994). Estimation of structural damping of buildings. *Proceedings of the XII* ASCE Structures Congress, Atlanta, GA, 1012-1017
- [12] Satake, N., Suda, K. I., Arakawa, T., Sasaki, A., & Tamura, Y. (2003): Damping evaluation using full-scale data of buildings in Japan. *Journal of structural engineering*, **129** (4), 470-477.
- [13] Beck, J., and P. Jennings (1980): Structural identification using linear models and earthquake records. *Earthquake Engineering & Structural Dynamics* 8 (2), 145-160.
- [14] Cruz, C., and E. Miranda (2016): Evaluation of damping ratios for the seismic analysis of tall buildings. *Journal of Structural Engineering*, doi:10.1061/(ASCE)ST.1943-541X.0001628, 1-10
- [15] USGS, CGS, & ANSS (2014): Center for engineering strong motion data. http://strongmotioncenter.org> (Oct. 18, 2014)