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# PLANAR COMPRESS WAVE SCATTERING AND ENERGY DIMINUTION DUE TO RANDOM INHOMOGENEITY OF MATERIAL DENSITY

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## *Abstract*

Slender engineering structures are endangered by a surface seismic wave. Subsoil parameters include a significant random inhomogeneity, which strongly influences character, spectral composition and intensity of such waves. Therefore, the seismic shock in a certain distance from the epicenter changes its degree and type, depending on a random character of the subsoil parameters in the positive but also in the negative meaning of the term. The paper investigates the compress wave propagation in two-dimensional continuum, the density of which is a continuous random function of x,y coordinates. This basic concept refers materials with in-homogeneity, which are encountered in subsoil being attacked by seismic wave. The random in-homogeneity influences significantly the wave structure and energy diffusion due to scatter at in-homogeneities. The density consists of a constant mean value and density fluctuations. Relevant centered two-dimensional random process is considered to be Gaussian, stochastically homogeneous and ergodic in x,y coordinates. The integral spectral decomposition in space is employed to derive the governing integro-differential system. It includes unknown deterministic component of the response and functions characterizing its random part. An approximate supposition that Gaussian imperfections lead to non-centered Gaussian response is adopted. There is shown a steep drop of the response deterministic part and a simultaneous increase of the response uncertainty (stochastic part) with raising distance from the point of excitation. These processes don't represent any mechanical energy loss, but only changes of its form. An upper limit of the excitation frequency (critical frequency) depending predominantly on the mean correlation length of imperfections has been found. Some ideas of application in earthquake engineering are given.

*Keywords: Wave propagation, Material random in-homogeneity, Energy random scattering.*

## 1. Introduction

The surface seismic wave propagates from an epicenter area and endangers slender engineering structures in surroundings. The subsoil parameters include a significant degree of random inhomogeneity. Therefore, in the course of propagation through a non-homogeneous medium, the type of a seismic attack, which is originally more or less deterministic movement of soil, gradually disappears while the random component of the wave increases with the distance from the epicenter. In a certain distance from the epicenter the wave changes fully into the broad-band random process with strongly time dependent nonhomogeneous spectral density. Nevertheless, this spectral density is fictitious only, as the primary source is the wave passing through a certain point. The modeling initiated by the wave propagation reveals to be much more inherent than that working with the given spectral density which is difficult to be reliably determined for a certain point.

It is necessary to be aware that this transformed excitation can afflict the structure stronger or weaker depending on its spectral properties and a number of other factors (contact, nonlinearities, etc.). Materials like concrete, polycrystalline metals or composites with micro-inclusions and micro-fibers reinforcing can serve as of other examples. Although scale of these systems is significantly different the qualitative character is identical. Therefore, like in other domains of physics also seismic engineering should investigate the problem of wave propagation in a continuum with randomly variable parameters.

It should be highlighted that especially structures with dense eigen-frequences spectrum are very sensitive to broad-band excitation of the above type, as they appear as selective resonaters under this excitation (guyed masts, suspension bridges, etc.

From the viewpoint of dynamics, the paper outlines an analysis of compressional wave propagation in two-dimensional continuum, the density of which is a continuous random function of space coordinates.

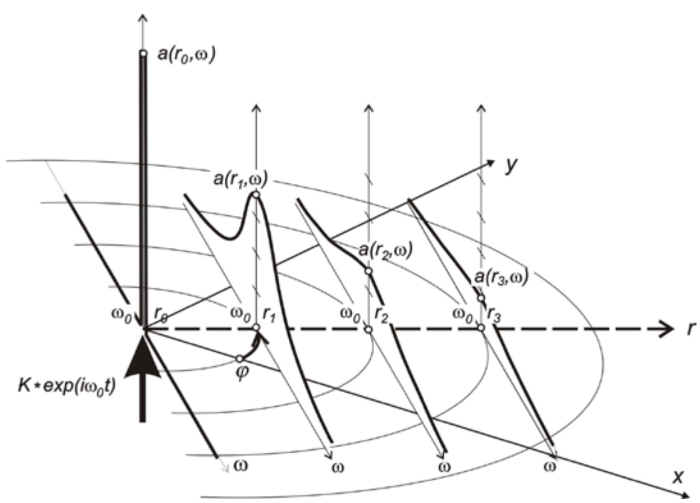


Fig. 1: Dropping of the response frequency curve with an increasing distance $|r - r_0|$, $r_0 = 0$ from the excitation point $r_0$; $\omega_0$- excitation frequency; $a(r, \omega)$- amplitude of the wave with frequency $\omega$.

Although the macroscopic mean values of physical parameters are commonly considered as constant, it is impossible to avoid the influence of random imperfections or nonhomogeneity in many cases. They originate from micro polycrystalline structure of metals, from micro-inclusions in composites with ceramic matrices, etc. The random variability of material density and elasticity parameters results in a stochastic component of the response, even if the excitation itself is deterministic. This phenomenon can be observed in experiments when parameters of induced wave motion in the excitation point are compared with the wave response at various distances from the source. In such a case the dispersion of the results is not determined merely by an unevenness of the experimental equipment, but also by the character of the material itself, see Fig. 1. For the harmonic excitation with a frequency $\omega_0$ in the origin, the response frequency curve in the excitation point itself can be plotted in the $(\Phi_0, \omega)$ plane as a Dirac-like function which is non-trivial in point $\omega_0$ only.

Two reasons of the response frequency curve dropping are obvious, when increasing distance r from the point of excitation: (i) geometric scatter in space leading to the wave amplitude diminution, however conserving non-trivial frequency portrait; (ii) due to random non-homogeneity of the material frequency curve changes into a certain "anthill" form in the ($\Phi_0, \omega$) plane containing wide range of frequencies.

In principle, as the value characterizing the deterministic part of the response is dropping with increasing r, the stochastic part of the response is rising, see Fig. 2. This figure originates from [1] and analogous results follow also from [2]. Parameters $|m_1(x)|$, $|m_{11}(x)|$ represent the mean value and the variance obtained for one dimensional semi-infinite rod with similar type of imperfections. Although the figure has the demonstration purpose only, analogous parameters are defined also in the current work being denoted: $\Psi_0(x, y)$, $h(\alpha, \beta, x, y)$ *or* $\Psi_0(r)$, $h(\rho, r)$. Their physical interpretation is slightly different due to analysis in two dimensional domain. For details, see Eqs (19), (20) in section three. Let us note that the drop of the deterministic part is not accompanied by any mechanical energy loss. Only its form is changed from deterministic into stochastic one so that no thermal energy is produced unless internal viscosity or other source of dissipation would be taken into account.

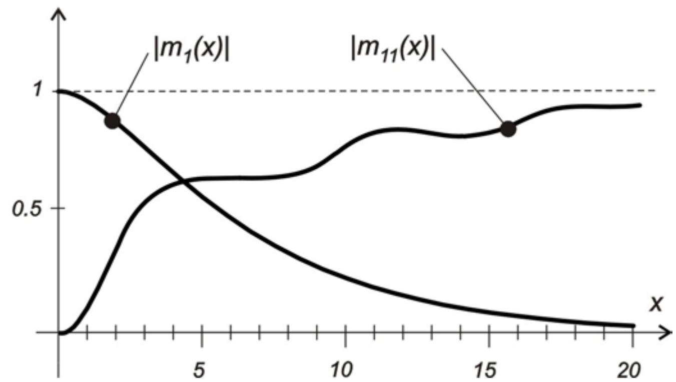


Fig. 2: Decay of the deterministic part $|m_1(r)|$ and an increase of the stochastic part $|m_{11}(r)|$ of the response with increasing distance from the excitation point; these parameters in further text are represented by stochastic parameters of the response: $\Psi_0(x, y)$, $h(\alpha, \beta, x, y)$ or $\Psi_0(r)$, $h(\rho, r)$, see Eqs (19), (20).

The rate of the response indeterminacy increases with increasing values of variance of material imperfection and with the rising excitation frequency. These two processes can lead to a critical state when the length of the propagating wave becomes comparable with the correlation length of imperfections. This state will manifest itself as a radical change of the response character. The problem will pass beyond the boundaries of stochastic mechanics and mathematical model loses its physical meaning. Similar effects can be observed in the FEM analysis, where a certain permissible upper boundary of the excitation frequency corresponds with the size and type of the element.

The increasing rate of the stochastic part of the response is not surprising. It fully corresponds with the law of the Boltzmann's entropy of probability which rises with indeterminacy rate reaching a maximum for the homogeneous Gaussian process in space coordinates.

During last two decades many papers appeared dealing with problem of wave propagation in a semi-infinite bar with material random imperfections, see [1] - [3] and many more. Various approaches have been used for investigation of the 1D problem, see for instance the integral spectral decomposition or analysis of the Fokker-Planck equation, e.g. [1], [2]. However, 2D and 3D approaches should be taken into consideration as a lot of problems in physics and technology cannot be satisfactorily analyzed using 1D models only.

The problem of wave propagation in stochastic medium attracted considerable attention in the past. Generally speaking, however, the works concerned merely certain qualitative estimates of the character of these processes in 2D and 3D media. An extensive survey of these activities with numerous references to further articles can be found in overview papers, for instance [5], [6] and many others. Special Issue of the Probabilistic Engineering Mechanics has been devoted to problems of materials with random non-homogeneities [7]. Of the other works directly connected with seismicity one can mention e.g. [8] - [11], etc. Other authors dealt with special problems arising in connection with various types of internal physical nonlinearity, e.g. [12] - [13] or investigated the influence of random roughness of reflection surface on wave scattering, [14]. With regard to the philosophy of construction of mathematical models, however, the above papers did not describe a number of effects specific for stochastic media. They were oriented to estimate the global properties of the response rather than to describe detailed properties of the wave propagating in a continuum with random fluctuations in material parameters. In general a lot of ideas can be found also in well founded monographs, e.g. [15], [16], [17], etc., dealing with stochastic differential equations including examples of application.

To enable a detailed analysis of motion, it is necessary to abandon mathematical models based on the small parameter method. This frequently used procedure does not characterize the motion in detail and, moreover, it gives physically contradictory results regarding the energy equilibrium law. The principal cause of this paradox is the fact that the small parameter method considers the random component of the response to be small and insignificant. This approach is acceptable only in the case of bodies with finite dimensions and very sparse spectrum of natural frequencies. In the domain of infinite dimensions, however, the stochastic part of the response becomes entirely dominant at a certain distance from the excitation point and the small parameter method is no longer convergent and provides meaningless results. It seems that this effect has many attributes from the viewpoint of stochastic as well as deterministic tasks. It has been discussed together with moving load [18], investigating chain-type structures e.g. [19], [20] or bending wave propagation, e.g. [21]. Regardless of various mathematical finesses and physical motivation observed in above references, the logical construction of the problem outlined above should start from the basic model of mass fluctuations encountered in common materials like metals, concrete, etc.

## 2. Material density imperfections

Material density is considered in the form which corresponds with a hypothesis that imperfections are relatively small related to mean value. So the following expression can be accepted:

$$\mu(x,y) = \mu_0 + \mu_\varepsilon(x,y), \tag{1}$$

$$\Im\{\mu_\varepsilon^2(x)\} << \mu_0; \tag{2}$$

$\mu_0$ - constant mean values of density;

$\mu_\epsilon(x,y)$ - continuous centered random Gaussian homogeneous process describing the fluctuations of density in variables $x$, $y$; the condition Eq. (2) express that the fluctuations are ”small” in comparison with the nominal value of $\mu_0$; however a very small but positive probability is admitted that this process may acquire a large absolute value in some points $x$, $y$, if this is the case in a very small neighborhood of these points; therefore Eq. 2 remains in force;

$\Im\{\cdot\}$ - operator of mathematical mean value with Gaussian density of probability.

Consequently, the respective correlation function is dependent on the distance of *(x, y)* and *(x’, y’)* points only and hence having a form of a surface of revolution:

$$K_{\mu\mu}(x,y) = K_\mu(r); \qquad r^2 = x^2 + y^2 \tag{3}$$

The correlation function cannot be introduced as the Dirac function taking random process as fully uncorrelated. This usually adopted hypothesis of perturbation independence in adjacent points of the region is unacceptable as it results in a meaningless violation of the energy equilibrium law. Indeed, referring the 1D case, see [1] or [4], the Dirac-like correlated imperfections lead to the response mean value with constant amplitude as it would correspond with the perfectly homogeneous material. Apparently the mean value becomes independent from any imperfections in materials. On the other hand, the random response component starts at zero in the excitation point, like in Fig. 2, and it is rising with a mild wavy character. Nevertheless, summing both energies results in a paradox that the energy amount is rising with the length coordinate. Moreover, it would mean that information inserted into the rod in excitation point is still unperturbed and without any tendency to be absorbed by a noise. Such a conclusion would also violate the Boltzman stochastic entropy principle, see well known monographs e.g. [15] or [16]. Nevertheless, it can be shown that the simplest acceptable model is the exponential correlation. Its application does not lead to any discrepancy with energy balance theorems and can be easily expressed mathematically. The meridian curve in 2D domain can be adopted in the same exponential form as in the 1D case:

$$K_{\mu\mu}(x,y)=\sigma_0^2\exp(-ar)=\sigma_0^2\exp(-a\sqrt{x^2+y^2});\;\; r\geq 0;\, a>0 \tag{4}$$

Let us determine the spectral density. Double Fourier transform can be outlined as follows:

$$S_{\mu\mu}(\alpha,\beta)=\sigma_0^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-a\sqrt{x^2+y^2}+\mathrm{i}\alpha x+\mathrm{i}\beta y)\,\mathrm{d}x\mathrm{d}y \tag{5}$$

To evaluate the double integral, the polar coordinates are to be used ($x = r\cos\varphi$, $y = r\sin\varphi$, $J = r$). Some modifications gives:

$$S_{\mu\mu}(\alpha,\beta)=\frac{\sigma_0^2}{4\pi^2}\int_0^{\infty}r\exp(-ar)\int_0^{2\pi}\exp(-\mathrm{i}r\sqrt{\alpha^2+\beta^2}\cos(\varphi-\varphi_0)\,\mathrm{d}\varphi\,\mathrm{d}r;$$
$$\cos\varphi_0=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}};\;\;\sin\varphi_0=\frac{\beta}{\sqrt{\alpha^2+\beta^2}} \tag{6}$$

The integrand is a function of φ with a period 2π. Then it holds that the integration interval can start in any arbitrary point if its length is kept being the whole period 2π. In such a case the phase shift $\varphi_0$ can be put equal to zero and Eq. (6) can be rewritten as follows:

$$S_{\mu\mu}(\alpha,\beta)=\frac{\sigma_0^2}{4\pi^2}\int_0^{\infty}r\exp(-ar)\int_0^{2\pi}\exp(-\mathrm{i}r\sqrt{\alpha^2+\beta^2}\cos\varphi\,\mathrm{d}\varphi\,\mathrm{d}r. \tag{7}$$

The internal integral represents the integral formulation of a cylindrical function $J_0$, so that the expression Eq. (7) gives:

$$S_{\mu\mu}(\alpha,\beta)=\frac{\sigma_0^2}{4\pi^2}\int_0^{\infty}r\exp(-ar)\cdot J_0(r\sqrt{\alpha^2+\beta^2})\,\mathrm{d}r. \tag{8}$$

The above integral enables to be expressed using $\Gamma(z)$ an $B(x, y)$ functions. Respecting their values in particular points, the spectral density can be finally written as follows:

$$S_{\mu\mu}(\alpha,\beta)=\frac{\sigma_0^2}{2\pi^2}\frac{a}{(a^2+\alpha^2+\beta^2)^{3/2}}=\frac{\sigma_0^2}{2\pi^2}\frac{a}{(a^2+\gamma^2)^{3/2}};\;\;(\gamma^2=\alpha^2+\beta^2). \tag{9}$$

The spectral density S $(\alpha,\beta)$ is a surface of revolution similarly like the correlation function Eq. (4).

## 3. Scalar potential of compress wave in two dimensions

The study is limited to an analysis of a compress wave when the wave enables to be described by one scalar potential $\Phi$ in $x$, $y$ coordinates. Excitation is considered as deterministic having a kinematic character and acting in the origin only. Its time history is supposed with a fixed frequency $\omega$, see Fig. 1.

The respective potential satisfies the Helmholtz equation:

$$\Delta\Phi(x,y,t)-\kappa^2\cdot\mu(x,y)\ddot{\Phi}(x,y,t)=0. \tag{10}$$

The problem formulation in 2D prevents immediate application of some popular solution tools. However, the procedure of the integral spectral decomposition seems to be still applicable despite of well known shortcomings. Using this method, the relevant integro-differential system can be inferred from the Eq. (10). The system includes two unknown deterministic components representing weak solution of the Eq. (10) in stochastic meaning: (i) mathematical mean value representing deterministic component; (ii) function characterizing random part of the response or propagating wave.

Let us we start considering kinematic harmonic excitation concentrated in the domain Ω:

$$\Phi(x,y,t)|_{\Omega} = \psi(x,y)\exp(\mathrm{i}\omega t). \tag{11}$$

where $\psi(x, y)$ is the complex amplitude of kinematic excitation in the domain Ω;

Domain Ω is supposed to be an infinite line coincident with $x$ axis or a point of coordinates origin. The former one produces wave propagating predominantly in y direction, while the latter one leads to radial wave.

Due to linear character of the problem the response can be written as mono-chromatic in the whole domain with phase shift dependent on the position $x, y$:

$$\Phi(x,y,t) = \Phi(x,y)\exp(\mathrm{i}\omega t). \tag{12}$$

Substituting expression Eq. (12) into Eq. (10) and regarding Eqs (1,2) we shall obtain a differential equation for the complex amplitude of the potential $\Psi(x, y)$:

$$\Delta\Psi(x,y) + \kappa^2 \cdot \omega^2 (\mu_0 + \mu_\varepsilon(x,y))\,\Psi(x,y) = 0, \tag{13}$$

where potential $\Psi(x, y)$ must comply with Sommerfeld condition for $x, y \rightarrow \infty$ and on the domain Ω it must hold:

$$\Psi(x,y)|_{\Omega} = \psi(x,y). \tag{14}$$

Provided the characteristics of the stochastic process $\mu_\epsilon(x, y)$ are available, it is possible to express the amplitude of the response $\Psi(x, y)$ in the form of the continuous stochastic process or its characteristics. In the meaning of the method of integral spectral decomposition we shall write the fluctuating parts of the mass Eq. (1) as a Stieltjes integral in both coordinates:

$$\mu_\varepsilon(x,y) = \int_{-\infty}^{\infty}\!\!\int \exp(i(\alpha x + \beta y))\,\mathrm{d}M(\alpha,\beta), \tag{15}$$

where $\mathrm{d}M(\alpha,\beta)$ is a spectral differential of density fluctuations. The response, or its amplitude, can be written, in accordance with the adopted method of solution, in analogous form:

$$\Psi(x,y) = \Psi_0(x,y) + \Psi_\varepsilon(x,y) = \Psi_0(x,y) + \int_{-\infty}^{\infty}\!\!\int h(\alpha,\beta,x,y)\,\mathrm{d}P(\alpha,\beta). \tag{16}$$

where:

$\mathrm{d}P(\alpha\,\beta)$ - spectral differential of the response;
$\Psi_0(x, y)$ - mathematical mean value of the response;
$\Psi_\epsilon(x, y)$ - fluctuation part of the response corresponding with imperfections of the mass;
$h(\alpha, \beta, x, y)$ - unknown deterministic functions acting as modulations of the process $\mathrm{d}P(\alpha\,\beta)$.

Inserting the expressions Eqs (15), (16) in Eq. (13), following equation arises:

$$\begin{aligned}\Delta\Psi_0(x,y) + \kappa^2\omega^2\mu_0\,\Psi_0(x,y) + \int_{-\infty}^{\infty}\!\!\int[\Delta h(\alpha,\beta,x,y) + \kappa^2\omega^2\mu_0 h(\alpha,\beta,x,y)]\mathrm{d}P(\alpha,\beta) + \\ +\kappa^2\omega^2\Psi_0\int_{-\infty}^{\infty}\!\!\int \mathrm{e}^{\mathrm{i}(\alpha x+\beta y)}\,\mathrm{d}M(\alpha,\beta) + \kappa^2\omega^2\int\!\!\int\int_{-\infty}^{\infty}\!\!\int \mathrm{e}^{\mathrm{i}(\alpha x+\beta y)} h(\alpha',\beta',x,y)\mathrm{d}P(\alpha',\beta')\mathrm{d}M(\alpha,\beta) = 0.\end{aligned} \tag{17}$$

The process $\Psi\epsilon(x, y)$ is not strictly Gaussian. In the actual case, however, it does not differ much from that (only slight asymmetry, somewhat larger and variable excess); consequently, it may be taken approximately as Gaussian and centered process. That means that its description requires only two statistical moments (mathematical mean value, correlation function or variance) as functions of coordinates $x, y$. The mathematical mean value may be considered as the deterministic (effective) part of the response, while the correlation function describes its random part. The same can be adopted concerning processes $\mathrm{d}M(\alpha\,\beta)$, $\mathrm{d}P(\alpha\,\beta)$. Furthermore, processes $\mathrm{d}M(\alpha\,\beta)$, $\mathrm{d}P(\alpha\,\beta)$ can be regarded as statistically orthogonal so that the Wiener-Khinchin relations, see, e.g. [15] or [17], can be applied:

$$\Im\{\overline{\mathrm{d}M}(\alpha,\beta)\,\mathrm{d}P(\alpha',\beta')\} = S_{\mu\Psi}(\alpha,\beta)\,\delta(\alpha-\alpha')\,\delta(\beta-\beta')\,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{18}$$

where $\delta(\cdot)$ is the Dirac delta function. Provided the condition Eq. (2) is complied with, it is possible to neglect the terms with the second derivatives in the derivation of the products of random processes. The operator of mathematical mean value should be applied now to Eq. (17). With regard to the properties of the functions $\Psi_0(x, y)$, $h(\alpha, \beta, x, y)$ the operator of integration and the operator of mathematical mean value are mutually interchangeable. Thus with respect to Eq. (18) we obtain the equation:

$$\Delta\Psi_0(x,y)+\kappa^2\omega^2\mu_0\,\Psi_0(x,y)+\kappa^2\omega^2\int_{-\infty}^{\infty}\int \mathrm{e}^{-\mathrm{i}(\alpha x+\beta y)}h(\alpha,\beta,x,y)\,S_{\mu\Psi}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta=0. \tag{19}$$

The integral term in Eq. (19) extends the classical equation by the influence of random imperfections. Consequently, the deterministic part of the response is influenced significantly by the fluctuations of the mass $\mu_\varepsilon(x, y)$. The theory based on the small parameter method results in an equation in which this integral term is missing and comes to the conclusion that the mathematical mean value of response equals the classical solution. Consequently, the question arises: what is the source of energy of the fluctuating part of the response which rises from zero in excitation point (0, 0). From this viewpoint the integral term in Eq. (19) is of principal significance.

Should we multiply Eq. (17) by the spectral differential $\overline{\mathrm{d}M(\alpha,\beta)}$ before the mathematical mean operator is applied, several modifications lead to the equation:

$$(\Delta h(\alpha,\beta,x,y)+\kappa^2\omega^2\mu_0 h(\alpha,\beta,x,y))\,\mathrm{e}^{-\mathrm{i}(\alpha x+\beta y)}\,S_{\mu\Psi}(\alpha,\beta)+\omega^2 S_{\mu\mu}(\alpha,\beta)\Psi_0(x,y)=0, \tag{20}$$

where it has been denoted:

$S_{\mu\mu}(\alpha, \beta)$ - spectral density of the process $\mu_\epsilon(x, y)$ characterizing random variability of the mass.
$S_{\mu\Psi}(\alpha, \beta)$ - cross-spectral density of $\mu_\epsilon(x, y)$ and $\Psi_\epsilon(x, y)$; it is closely related with unknown $h(\alpha,\beta,x, y)$.

Eqs (19) and (20) make up integro-differential system with two unknowns $\Psi_0(x, y)$, $h(\alpha, \beta, x, y)$. It can be shown for a mono-chromatic wave that a deterministic boundary condition exists on the domain Ω. It represents a mean value of kinematic excitation Eq. (11):

$$\Im\{\Psi(x,y)\}|_\Omega=\Psi_0(x,y)|_\Omega=K\cdot f(y). \tag{21}$$

With respect to a deterministic character of the kinematic excitation in the domain Ω, all stochastic moments of the response should vanish here indicating the zero rate of indeterminacy. So it means:

$$\Psi_\varepsilon(x,y)|_\Omega=0 \quad\Rightarrow\quad h(\alpha,\beta,x,y)|_\Omega=0. \tag{22}$$

Herewith the problem of the stochastic differential equation Eq. (17) has been transformed into the system of two deterministic differential equations (19), (20) for unknowns, which represent together with conditions (21), (22) the random character of the system response.

## 4. Excitation along the *y* axis

Solution of the system Eqs (19), (20) can be written with regard to expected form of the spectral density $S_{\mu\mu}(\alpha,\beta)$:

$$\Psi_0(x,y)=\sum_{k=1}^{n}C_k e^{\lambda_k x}\cdot f(y),\quad h(\alpha,\beta,x,y)=\sum_{k=1}^{n}H_k(\alpha,\beta)\,e^{(\lambda_k x+\mathrm{i}\alpha)x+\mathrm{i}\beta y}\cdot f(y). \tag{23}$$

The upper limit $n$ conforms to the type of the function $S_{\mu\mu}(\alpha, \beta)$. Function $f(y)$ corresponds with the distribution of the excitation amplitude along the y axis, see Eq. (11). In principle the solution could be formulated as Fourier transform with respect to $y$ axis. Thus eliminating this coordinate the further strategy is similar with that keeping the amplitude constant. Hence on the level of qualitative analysis function $f(y)$ can be taken constant: $f(y) = 1$.

Series in Eqs (23) shall be introduced into Eqs (19), (20), which yields:

$$\begin{aligned} &C_k \cdot (\lambda_k^2 + \omega^2\kappa^2\mu_0) + \omega^2\kappa^2 \int_{-\infty}^{\infty}\int H_k(\alpha,\beta)\, S_{\mu\Psi}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta = 0, \\ &H_k(\alpha,\beta)\cdot((\lambda_k + \mathrm{i}\alpha)^2 - \beta^2 + \omega^2\kappa^2\mu_0)\, S_{\mu\Psi}(\alpha,\beta) + C_k\omega^2 S_{\mu\mu}(\alpha,\beta) = 0. \end{aligned} \tag{24}$$

Eliminating $C_k$ and $H_k(\alpha, \beta)$, following characteristic (integro-algebraic) equation can be carried out:

$$(\lambda_k^2 + \omega^2\kappa^2\mu_0) + \omega^4\kappa^2 \int_{-\infty}^{\infty}\int \frac{S_{\mu\mu}(\alpha,\beta)\mathrm{d}\alpha\,\mathrm{d}\beta}{(\lambda_k + \mathrm{i}\alpha)^2 - \beta^2 + \omega^2\kappa^2\mu_0} = 0. \tag{25}$$

Influence of stochastic component of the mass density in Eq. (25) is well visible being described by its spectral density $S_{\mu\mu}$ in the integral term. Using formula Eq. (9) obtained in the previous part, Eq. (25) can be rewritten in the form:

$$(\lambda_k^2 + \omega^2\kappa^2\mu_0) - \frac{\omega^4\kappa^2\sigma_0^2}{2\pi} \int_{-\infty}^{\infty}\int \frac{a}{(a^2+\alpha^2+\beta^2)^{3/2}} \cdot \frac{\mathrm{d}\alpha\,\mathrm{d}\beta}{(\lambda_k + \mathrm{i}\alpha)^2 - \beta^2 + \omega^2\kappa^2\mu_0} = 0. \tag{26}$$

Transformation into the polar coordinates $\rho$, $\theta$: ($\alpha = \rho\cos\theta$, $\beta = \rho\sin\theta$, $J = \rho$, $\alpha^2 + \beta^2 = \rho^2$) yields a modified form which reads:

$$(\lambda_k^2 + \omega^2\kappa^2\mu_0) - \frac{\omega^4\kappa^2\sigma_0^2}{2\pi} \int_{-\infty}^{\infty} \frac{a\rho}{(a^2+\rho^2)^{3/2}} \int_0^{2\pi} \frac{\mathrm{d}\rho\,\mathrm{d}\theta}{\lambda_k^2 + \omega^2\kappa^2\mu_0 - \rho^2 + 2\mathrm{i}\rho\lambda_k\cdot\cos\theta} = 0. \tag{27}$$

The internal integral admits to be evaluated in a closed form using $\Gamma(z)$ and $B(x, y)$ functions:

$$(\lambda_k^2 + \omega^2\kappa^2\mu_0) - \frac{\omega^4\kappa^2\sigma_0^2}{2\pi} \int_0^{\infty} \frac{a\rho}{(a^2+\rho^2)^{3/2}} \cdot \frac{2\cdot B(\frac{1}{2},\frac{1}{2})\cdot\mathrm{d}\rho}{((\lambda_k^2 + \omega^2\kappa^2\mu_0 - \rho^2)^2 + 4\rho^2\lambda_k^2)^{1/2}} = 0. \tag{28}$$

Inserting particular values, the characteristic equation gains an integro-algebraic form:

$$(\lambda_k^2 + \omega^2\kappa^2\mu_0) - \frac{\omega^4\kappa^2\sigma_0^2}{2\pi}\cdot I_k = 0, \quad I_k = \int_0^{\infty} \frac{a\rho\cdot\mathrm{d}\rho}{[(a^2+\rho^2)^3((\lambda_k^2 + \omega^2\kappa^2\mu_0 - \rho^2)^2 + 4\rho^2\lambda_k^2)]^{1/2}}. \tag{29}$$

The integral in Eq. (29) should be investigated in detail. The respective integrand is an irrational function and hence the integration using various versions of methods based on theory of holomorphic and multivalent functions in Gaussian plane are rather non-applicable. Nevertheless, the integral Eq. (29b) obviously exists whatever are parameters $a$, $\lambda_k^2$, $\omega^2\kappa^2\mu_0$. It is continuous and positive on the whole interval including point $\rho = 0$. Starting at a certain $\rho = \rho_0$ it can be sufficiently approximated by a function $\rho^{-4}$, which is integrable on the interval $\rho \in (\rho_0, \infty)$.

## 5. Excitation concentrated in point (0, 0)

Let us have a look at the system Eqs (19), (20), when the harmonic excitation is concentrated only in the coordinates origin. In such a case respecting structure of the spectral density $S_{\mu\mu}(\alpha, \beta)$ according to Eq. (9), the solution can be suggested in the form:

$$\begin{aligned} &\Psi_0(r) = \sum_{k=1}^{n} C_k J_0(\lambda_k r), \\ &h(\rho, r) = \sum_{k=1}^{n} H_k(\rho)\, J_0^*(\lambda_k r), \text{where} \quad J_0^*(\lambda_k r) = J_0(\lambda_k r + \rho r). \end{aligned} \tag{30}$$

The structure of unknowns Eqs (30) complies with the monopole excitation type. Provided the point excitation in the origin has a form of the multipole, then following expressions should be employed:

$$\begin{aligned}\Psi_0^l(r) &= \sum_{k=1}^{n} C_k J_l(\lambda_k r)\cdot\cos l\varphi,\\ h^l(\rho,r) &= \sum_{k=1}^{n} H_k(\rho)\, J_l^*(\lambda_k r)\cdot\cos l\varphi,\end{aligned} \tag{31}$$

where $J_l(\lambda_k r)$ - Bessel function of the order $l \geq 0$.

Similarly like in the previous section, series in Eqs (30) should be inserted into Eqs (19), (20). Taking into account relations:

$$\frac{\mathrm{d}^2 J_0}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\,J_0}{\mathrm{d}r}=-\lambda_k^2 J_0\,,\qquad \frac{\mathrm{d}^2 J_0^*}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\,J_0^*}{\mathrm{d}r}=-(\lambda_k+\rho)^2\,J_0^*\,, \tag{32}$$

several modifications bring us to the system:

$$\begin{aligned}C_k\cdot(-\lambda_k^2+\omega^2\kappa^2\mu_0)J_0+\omega^2\kappa^2\int_{-\infty}^{\infty}\int \mathrm{e}^{-i(\alpha x+\beta y)}H_k(\rho)\,J_0^*\cdot S_{\mu\Psi}(\rho)\,\mathrm{d}\alpha\,\mathrm{d}\beta=0,\\ H_k(\rho)\cdot((-\lambda_k+\rho)^2+\omega^2\kappa^2\mu_0)\,\mathrm{e}^{-i(\alpha x+\beta y)}J_0^*\cdot S_{\mu\Psi}(\rho)+C_k\omega^2 J_0 S_{\mu\mu}(\alpha,\beta)=0.\end{aligned} \tag{33}$$

Parameters $C_k$, $H_k$ and functions $J_0$, $J_0^*$, $\mathrm{e}^{-\mathrm{i}(\alpha x+\beta y)}$, $S_{\mu\Psi}(\rho)$ can be eliminated. These steps enable to obtain the characteristic equation:

$$(-\lambda_k^2+\omega^2\kappa^2\mu_0)-\omega^4\kappa^2\int_{-\infty}^{\infty}\int\frac{S_{\mu\mu}(\rho)\mathrm{d}\alpha\,\mathrm{d}\beta}{-(\lambda_k+\mathrm{i}\rho)^2+\omega^2\kappa^2\mu_0}=0. \tag{34}$$

The spectral density following Eq. (9) should be put into Eq. (34). Transformation of the double integral term into polar coordinates similarly like in the previous part enables to reformulate the previous equation:

$$(\lambda_k^2+\omega^2\kappa^2\mu_0)-\frac{\omega^4\kappa^2\sigma_0^2}{2\pi}\int_{-\infty}^{\infty}\frac{a\rho}{(a^2+\rho^2)^{3/2}}\int_0^{2\pi}\frac{\mathrm{d}\rho\,\mathrm{d}\theta}{\lambda_k^2+\omega^2\kappa^2\mu_0-\rho^2+2\mathrm{i}\rho\lambda_k\cdot\cos\theta}=0. \tag{35}$$

Once again, the internal integral can be solved in a closed form using $\Gamma(z)$ and $B(x, y)$ functions. Performing additional modifications, we get the final form of the characteristic equation:

$$-\lambda_k^2+\omega^2\kappa^2\mu_0-\frac{\omega^4\kappa^2\sigma_0^2}{2\pi}\cdot I_k=0,\qquad I_k=\int_0^{\infty}\frac{a\rho\cdot\mathrm{d}\rho}{[(a^2+\rho^2)^3((\lambda_k^2+\omega^2\kappa^2\mu_0-\rho^2)^2+4\rho^2\lambda_k^2)]^{1/2}}. \tag{36}$$

Analysis of the characteristic equation Eq. (36) shows that for very small variance $\sigma_0^2$ and low frequency $\omega^2$ the $\lambda_k^2$ is negative which implicates that the solution itself is periodic. When both are increasing, then influence of above elliptic integral is rising, complicated algebraic equation arises and adequate investigation should be done to separate solutions of the characteristic equation leading to periodic, non-periodic and beating type solutions. However, roughly speaking, from the qualitative point of view, the first Eq. (36) shows that negative solution can be obtained when it holds:

$$\omega^2\cdot\sigma_0^2<\mu_0\,/\,I_k\,. \tag{37}$$

Therefore, it exists a certain upper limit of frequency $\omega$ (or lower limit of the wave length) beyond which, with respect to imperfection variance, the physical interpretation of results cannot be accepted, although mathematical solution still exists. Cases like that should be investigated using more sophisticated mathematical models.

## 6. Analysis of the integral term in the characteristic equation

Integrals (29b) and (36b) represent influence of random fluctuations of mass density $\mu_\varepsilon$. It is evident that $\sigma_0^2$ approaching zero leads to the classical characteristic equations. In such a case their roots are producing solutions conserving mechanical energy without any dispersion. Indeed, Eq. (28) provides $\lambda_k$ as a pure imaginary couple and therefore Eq. (23a) represents a non-damped wave whose amplitude is driven in *y* direction by mean of the function *f (y)*. Concerning radial wave outlined by Eq. (30a) its basic dropping tendency with increasing *r* is given by advancing of the wave front on circles with increasing radius. This fact is mathematically expressed by $J_0$ ($\lambda_k$) function having an alternating character with decreasing amplitudes. Therefore, the integral Eq. (36b) implicates even faster amplitude decay which consists of two sources. The first one is classical being given by energy spread due to increasing diameter of the wave front and the second one is due to fluctuations of mass density as it is characterized by the integral in question. Because the amplitude drop of a cylindrical function with rising order is faster, consequently also the energy spread in space accelerates with increasing parameter *l*.

Comparing Eqs (29b) and (36b) we can see, that their structure is in general identical. Hence we can discuss simultaneously both of them. The structure of Eq. (29b) and Eq. (36b) suggests, that the integral can be expressed in a closed form using elliptic integrals.

Transformation $\rho^2 \to \tau$ and several modifications lead to the expression as follows:

$$I_k = \int_0^\infty \frac{a\,\mathrm{d}\tau}{[(\tau - A)^3((\tau - B)^2 + C)]^{1/2}}, \tag{38}$$

Where:

$$A = -a^2, \quad B = \kappa^2 \mu_0 \omega^2 - \lambda_k^2, \quad C = 4\kappa^2 \lambda_k^2 \mu_0 \omega^2 .$$

According to [22], case 241.03, integral (38) can be expressed using the cosine amplitude cn (Jacobian elliptic function) in the form of:

$$I_k = \int_0^u \frac{1 - \mathrm{cn}(u\,|\,\eta)}{1 + \mathrm{cn}(u\,|\,\eta)}, \tag{39}$$

where $\eta$ is modulus of the Jacobian function and $\phi$ is the amplitude:

$$\eta = \frac{\zeta - A + B}{2\zeta}, \quad \zeta = \sqrt{(B - A)^2}, \quad \mathrm{cn}(u\,|\,\eta) = \cos(\phi) = -\frac{\zeta + A}{\zeta - A}.$$

The expression (39) can be integrated as follows:

$$I_k = \frac{1}{\zeta^{3/2}}\left(\sqrt{\frac{4\zeta}{-A}[1 + \eta((\frac{\zeta + A}{\zeta - A})^2 - 1)]} - 2E(\phi\,|\,\eta) + u_1\right), \tag{40}$$

where $E(\phi\,|\,\eta)$ stands for the elliptic integral of the second kind:

$$E(\phi\,|\,\eta) = \int_0^\phi \sqrt{1 - \eta \sin^2 \vartheta}\,\mathrm{d}\vartheta .$$

The integral is well defined for real negative *A* and complex coefficients *B*,*C*. For the real coefficients *B*,*C* is the value of $I_k$ real and positive. In other words, as a function of the variable $\lambda_k$ is the integral continuous function.

The both equations (29a) and (36a) should be solved for $\lambda_k$ which will be further substituted into Eqs (23) or (30). According to the parameters of the system the resulting roots will be either complex pairs or real values. The roots will be real for values of parameter $\omega^4 \sigma_0^2$ larger than certain value. In this case the presented model losses its physical meaning and should not be used.

The explicit formula (40) is unfortunately rather complicated and does not provide any significant insight into the structure of solution of (29a) or (36a). A convenient numerical procedure for searching of the roots can use the slightly perturbed imaginary couple from the deterministic case (when $\sigma_0^2$ = 0) as an initial approximation. Preliminary numerical experiments have shown usefulness of this approach.

## 7. Conclusion

The aim of the paper is to establish a theoretical background for an assessment of engineering structures attacked by a seismic wave of a certain initial intensity. This new concept is based on a seismic wave propagation in surrounding subsoil which transforms the wave into the random process due to stochastic character of surrounding continuum.

In particular, the paper attempts for an analysis of compress wave propagation in two-dimensional continuum, the density of which is a continuous random function of $x, y$ coordinates. This basic concept refers to materials with microscopic non-homogeneity (concrete, poly-crystalline metals, etc.). It follows-up several papers dealing with problem of wave propagation in a semi-infinite bar with material random imperfections using the method of integral spectral decomposition or Markov processes together with the Fokker-Planck equation.

The study is limited to an analysis of a compress wave. This wave enables to be described by one scalar potential in x, y coordinates being governed by Helmholtz equation. Excitation time history is considered as deterministic having a harmonic form with a fixed frequency $\omega$. Two geometric distributions are discussed: (i) continuous amplitude of the excitation distributed along the whole axis $y$; waves propagate predominantly in $x$ direction towards $\pm\infty$; (ii) excitation concentrated in the point (0, 0); cylindrical wave propagates in the interval $r \in (0, \infty)$; monopole as well as multipole variants are investigated.

Material density is considered as a sum of a constant mean value and material density fluctuations. Fluctuations are defined by the random function in a plane $(x, y)$. The centered two-dimensional random process is considered to be Gaussian, stochastically homogeneous and ergodic in both coordinates $x$, $y$. The variance of fluctuations is supposed to be small with respect to constant mean value of the mass density. The respective correlation function is dependent on the distance of two points only having a form of a surface of revolution. It has been shown that the correlation function cannot be introduced as the Dirac function taking random process as fully non-correlated. This hypothesis would result in a meaningless energy equilibrium law violation. The simplest acceptable model seems to be the exponential correlation. Its application does not lead to any discrepancy with energy balance theorems and can be easily expressed mathematically. The meridian curve in 2D domain can be adopted in the same exponential form as in the 1D case. Double Fourier transform provides the corresponding spectral densities in coordinates $\alpha$, $\beta$. The result is a function of revolution having irrational character.

The problem formulation in 2D prevents immediate application of some popular solution tools. However, the procedure of the integral spectral decomposition seems to be still applicable despite of well known shortcomings. Using this method, the relevant integro-differential system has been inferred from the Helmholtz equation. The system includes unknown deterministic components of the response: mean value and function characterizing the second central moment of the random part. An approximate supposition that the Gaussian imperfections lead to non-centered Gaussian response has been adopted. This approximation justifies that two stochastic moments are satisfactory for the full description of the random response.

There has been shown a steep drop of the deterministic part of the response due to material stochastic character and a simultaneous increase of the response uncertainty (stochastic part) with the growing distance from the axis or point of a deterministic (harmonic) excitation. Process of determinacy drop is very slow in a close neighborhood of the excitation domain. In a long distance from excitation source nearly the whole amount of energy has the random character which corresponds on the other hand with the law of the entropy increase. These processes do not represent any mechanical energy loss, but only changes of its form. An upper limit of the excitation frequency (critical frequency) depending predominantly on the mean correlation length of imperfections has been identified. It can be taken as an applicability limit of the mathematical model adopted.

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